The scaling limit of a critical random directed graph

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Abstract:
We consider the random directed graph $\vec{G}(n,p)$ with vertex set $\{1,2,\ldots,n\}$ in which each of the $n(n-1)$ possible directed edges is present independently with probability p. We are interested in the strongly connected components of this directed graph. A phase transition for the emergence of a giant strongly connected component is known to occur at $p = 1/n$, with critical window $p = 1/n + \lambda n^{-4/3}$ for $\lambda \in (-\infty, \infty)$. We show that, within this critical window, the strongly connected components of $\vec{G}(n,p)$, ranked in decreasing order of size and rescaled by $n^{-1/3}$, converge in distribution to a sequence $(\mathcal{C}_1, \mathcal{C}_2, \ldots)$ of finite strongly connected directed multigraphs with edge lengths which are either 3-regular or loops. The convergence occurs the sense of an ℓ^1 sequence metric for which two directed multigraphs are close if there are compatible isomorphisms between their vertex and edge sets which roughly preserve the edge-lengths. Our proofs rely on a depth-first exploration of the graph which enables us to relate the strongly connected components to a particular spanning forest of the undirected Erdős–Rényi random graph $G(n,p)$, whose scaling limit is well understood. This is joint work with Christina Goldschmidt.