Hydrodynamic limit and fluctuations for a mean field opinion model

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based on a joint work with
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Dynamics of the model

- Each opinion is updated after a random time with distribution $\exp(1)$
- Transitions:

$$\eta_x \to \begin{cases} \eta_x^+ := \alpha \eta_x + (1 - \alpha) & \text{with probability } R^N \\ \eta_x^- := \alpha \eta_x & \text{with probability } 1 - R^N \end{cases}$$

Markov process in $[0,1]^N$ with generator

$$L_N f(\eta) = \sum_{x=1}^{N} \left[R^N \left(f\left(\eta_x^+\right) - f(\eta_x) \right) + \left(1 - R^N\right) \left(f\left(\eta_x^-\right) - f(\eta_x) \right) \right]$$

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Empirical measures

$$\mu^N(t) = \frac{1}{N} \sum_{x=1}^{N} \delta_{\eta_x(t)}(dr) \in \mathcal{M}^+([0,1])$$

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Initial distribution

Given $u_0 : \mathbb{R} \to \mathbb{R}^+$ a density profile such that supp $u_0 \subset [0,1]$. We suppose that

$$\left\langle \mu^N(0), G \right\rangle \xrightarrow[N \to +\infty]{P} \int_0^1 u_0(r) G(r) dr \qquad \forall G \in C([0,1], \mathbb{R}).$$

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- ▷ Hypothesis on α: $\alpha = 1 - \frac{1}{\sqrt{N}}$

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$$L_N \left\langle \mu^N, G \right\rangle = \sum_{x=1}^{N} \Big\{ R^N \frac{1}{N} \left[G\left(\eta_x^+\right) - G\left(\eta_x\right) \right] + \left(1 - R^N\right) \frac{1}{N} \left[G\left(\eta_x^-\right) - G\left(\eta_x\right) \right] \Big\}$$

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▷ We consider the time scale $\sqrt{N}t$

Hydrodynamic limit

Theorem

For all $T > 0$ *and* $G \in C^1([0,1], \mathbb{R})$ *it holds that*

$$\sup_{t\in[0,T]} \left| \left\langle \mu^N \left(\sqrt{N} t\right), G \right\rangle - \int_0^1 u(t,r)\, G(r)\, dr \right| \xrightarrow[N\to+\infty]{P} 0,$$

where $u(t,r)$ *is the solution of the following PDE*

$$\begin{cases} \frac{\partial}{\partial t} u(t,r) = (r - r_0) \frac{\partial}{\partial r} u(t,r) + u(t,r) \\ u(0,r) = u_0(r) \end{cases}$$

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▷ The system tends to a consensus state

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$$Q\left(M^N(t)\right) = \int_0^t \frac{1}{N^2}\left(\frac{1}{N}\sum_{x=1}^{N}\eta_x(s)^2 - 2\left(R^N(s)\right)^2 + R^N(s)\right)ds$$

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$$\Longrightarrow Q\left(M^N\left(N^2 t\right)\right) \simeq \int_0^t \left(R^N\left(N^2 s\right)\right)\left(1 - R^N\left(N^2 s\right)\right) ds$$

Theorem

For every $T > 0$*, the process* $\left\{R^N\left(N^2 t\right), t \in [0,T]\right\}$ *converges in distribution to the Wright-Fisher Diffusion, i.e. to the unique solution of the following SDE*

$$\begin{cases} dR\left(t\right) = \sqrt{R\left(t\right)\left(1 - R\left(t\right)\right)} dW\left(t\right) \\ R\left(0\right) = r_0 \end{cases}$$

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- $\tau = \inf_{t \geq 0} \{R(t) = 0 \text{ or } R(t) = 1\}$ is such that $\mathbb{E}(\tau) < +\infty$

Thank you for your attention!