Ergodicity of the KPZ Fixed Point

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The Kardar-Parisi-Zhang (KPZ) Universality Class

Universality Class for $1 + 1$ Stochastic Growth Models

- The universality class concept is an artifact of modern statistical mechanics that systemizes the idea that there are a few but important characteristics that determine the scaling behaviour of a stochastic model.
- In $1 + 1$ stochastic growth models the object of interest is a height function $h(x, t)$ over the one-dimensional substrate $x \in \mathbb{R}$ at time $t \geq 0$, whose evolution is described by a random mechanism.
- For fairly general models one has a deterministic macroscopic shape for the height function and its fluctuations, under proper space and time scaling, are expected to be characterized by a universal distribution.

The Kardar-Parisi-Zhang (KPZ) Universality Class

Universality Class for $1+1$ Stochastic Growth Models

- For instance, growth interfaces whose fluctuations are described by Gaussian statistics are said to be in the Gaussian universality class.
- In 1986, the existence of a new universality class was proposed by Kardar, Parisi and Zhang (KPZ) where the of stochastic growth evolution possesses a non-linear local slope dependent rate that.
- The KPZ equation, $\partial_t h = \frac{1}{2}(\partial_x h)^2 + \partial_x^2 h + \xi$, is a canonical example of such a growth model, providing its name to the universality class.

The Kardar-Parisi-Zhang (KPZ) Universality Class

Universality Class for $1+1$ Stochastic Growth Models

- In opposition to the Gaussian universality class, they predicted that the height function has fluctuations of order $t^{1/3}$, and on a scale of $t^{2/3}$ that non-trivial spatial correlation is achieved (KPZ scaling exponents).
- Illustrations of natural phenomena within this universality class include turbulent liquid crystals, bacteria colony growth and paper wetting, which are conjectured to converge under KPZ scaling to a universal space-time process $\mathfrak{h}_t(x)$, called the KPZ fixed point.

The Kardar-Parisi-Zhang (KPZ) Universality Class

Universality Class for $1+1$ Stochastic Growth Models

- The KPZ universality class became a notorious subject in the literature of physics and mathematics and, in the late nineties, a breakthrough was presented by Baik, Deift and Johansson (1999). (Exact formulas for the PNG model.)
- In the past twenty years there has been a significant amount of improvements of the theory. The exact statistics for certain initial geometries were computed using integrable models.
- A major step was achieved recently by Matetski, Quastel and Remenik (2017) using the totally asymmetric simple exclusion process (TASEP).

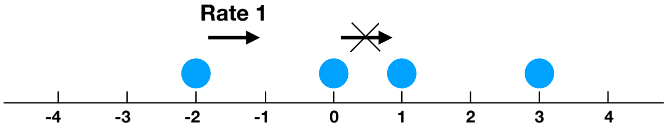
Totally Asymmetric Simple Exclusion Process

TASEP

- Markov process $(\eta_t\,,\ t \geq 0\,)$ with state space $\{0,1\}^{\mathbb{Z}}$.
- When $\eta_t(x) = 1$, we say that site x is occupied by a particle at time t, and it is empty if $\eta_t(x) = 0$.
- Particles jump to the neighbouring right site with rate 1 provided that the site is empty (the exclusion rule).

TASEP

Particles jump to the right with rate 1 provided the site is empty.



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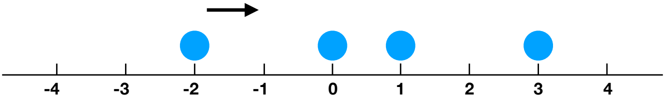

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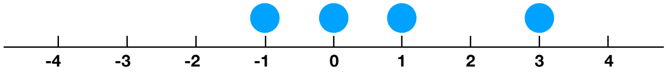

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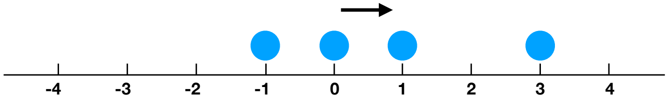

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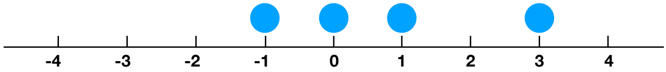

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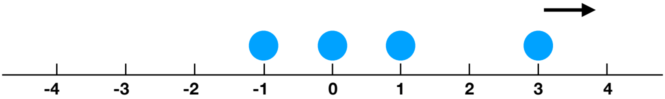

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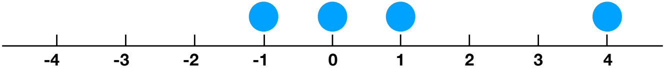


Interface Growth Model

TASEP Growth

Let N_t denote the total number of particles which jumped from site 0 to site 1 during the time interval $[0, t]$, and define

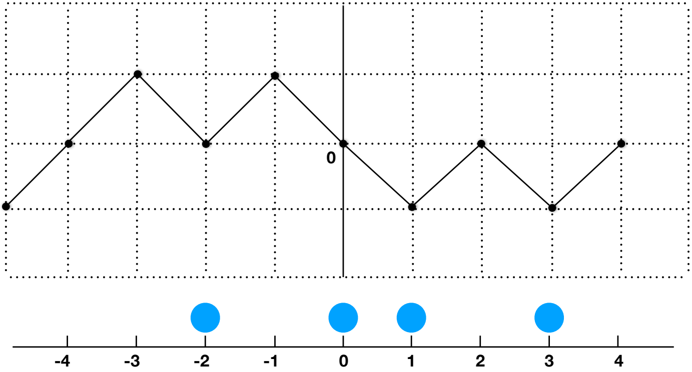
$$h_t(k) = \begin{cases} 2N_t + \sum_{j=1}^{k}(1 - 2\eta_t(j)) & \text{for } k \geq 1 \\ 2N_t & \text{for } k = 0 \\ 2N_t - \sum_{j=k+1}^{0}(1 - 2\eta_t(j)) & \text{for } k \leq -1\,. \end{cases}$$

Interface Growth Model

TASEP Growth

- Markov process $(h_t\ ,\ t \geq 0\ )$ with state space $\mathbb{Z}^{\mathbb{Z}}$.
- $h_t(k)$ is the value of height function at position $k \in \mathbb{Z}$ at time t.
- Local minimum becomes local maximum with rate 1.

TASEP Growth



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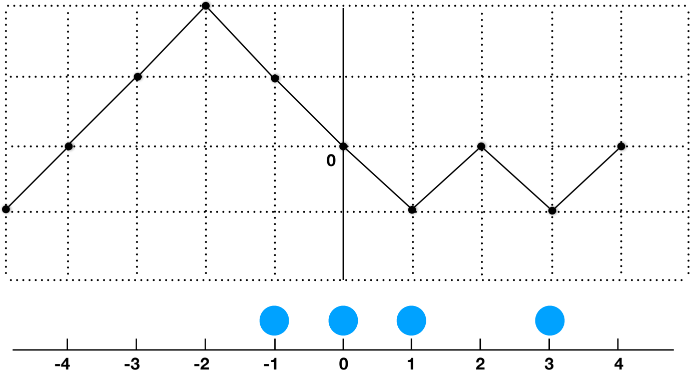

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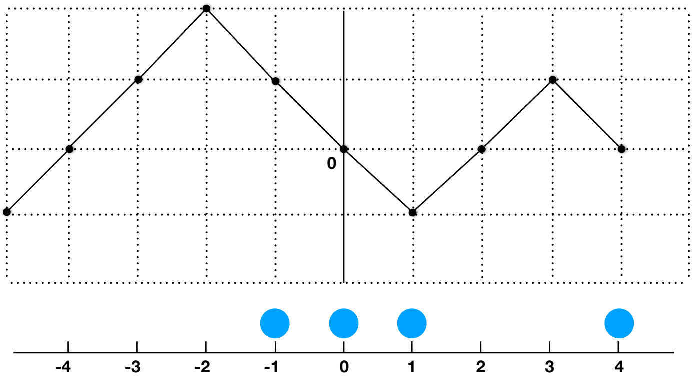

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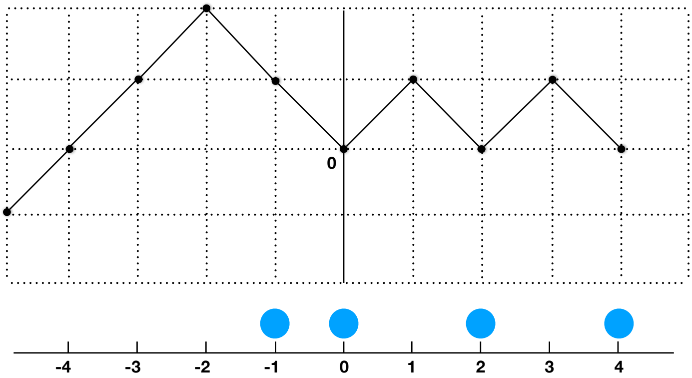

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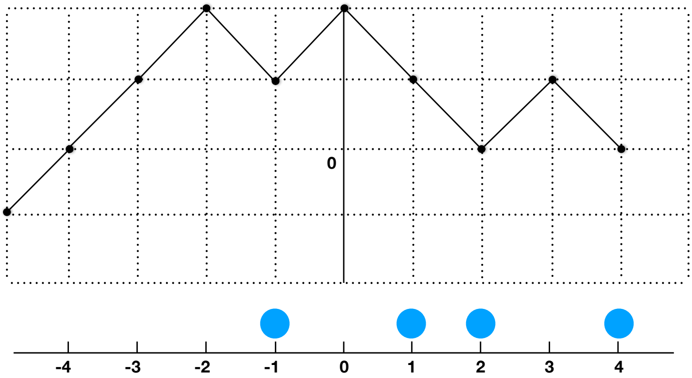

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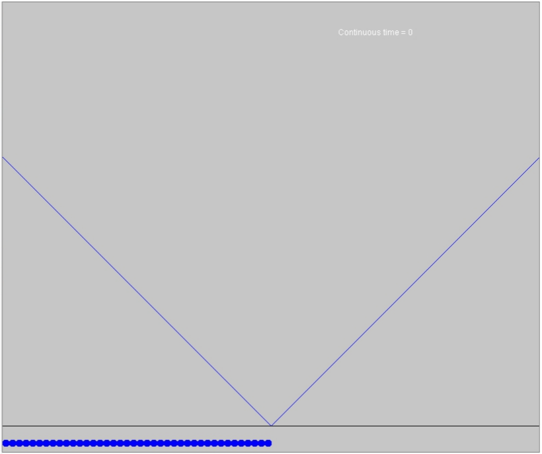

Figure: Narrow Wedge Initial Profile (Patrick Ferrari, Univ. Bonn).

TASEP Growth

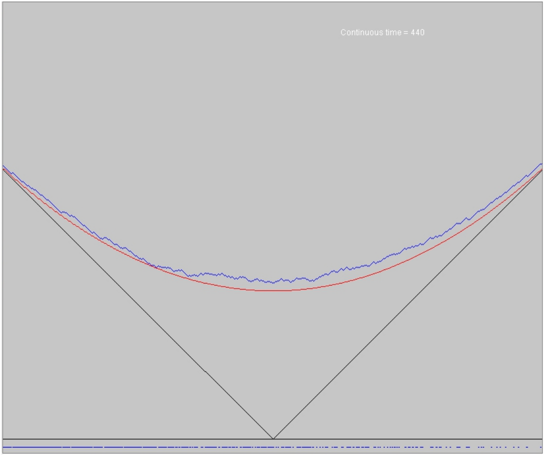

Figure: Narrow Wedge Initial Profile (Patrick Ferrari, Univ. Bonn).

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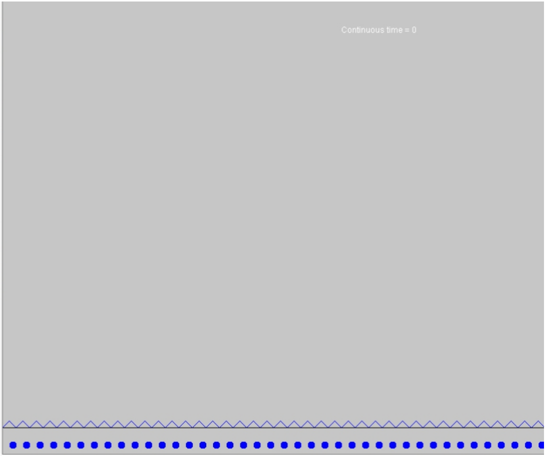

Figure: Flat Initial Profile (Patrick Ferrari, Univ. Bonn).

TASEP Growth

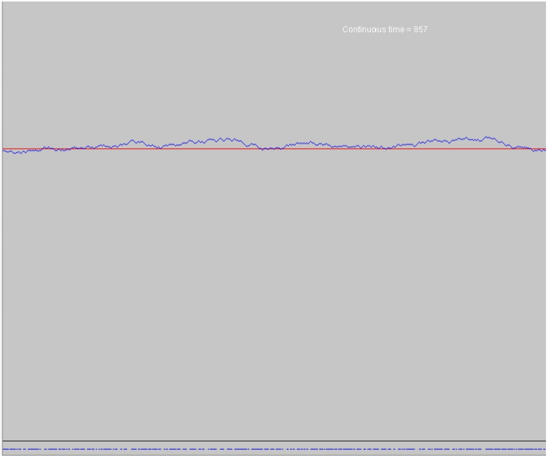

Figure: Flat Initial Profile (Patrick Ferrari, Univ. Bonn).

TASEP Growth

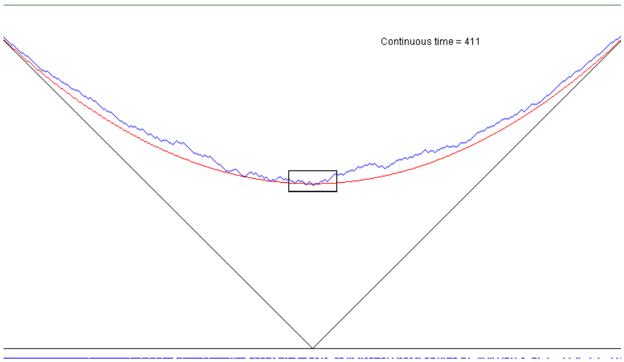

Figure: Scaling in a $n^{2/3} \times n^{1/3}$ rectangle.

TASEP Growth and the KPZ Fixed Point

Let

$$\mathfrak{h}_{n,t}(x) := \frac{tn - h_{2tn}^{(n)}\left(\lfloor 2xn^{2/3} \rfloor\right)}{n^{1/3}},$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in [-a, a] \subseteq \mathbb{R}$.

Theorem [Matetski, Quastel and Remenik '17]

If

$$\lim_{n\to\infty} \mathfrak{h}_{n,0}(\cdot) \stackrel{dist.}{=} \mathfrak{h}(\cdot),$$

then

$$\lim_{n\to\infty} \mathfrak{h}_{n,t}(\cdot) \stackrel{dist.}{=} \mathfrak{h}_t(\cdot;\mathfrak{h}),$$

where $(\mathfrak{h}_t(\cdot;\mathfrak{h}), t \geq 0)$ is the KPZ fixed point whit $\mathfrak{h}_0 = \mathfrak{h}$.

The KPZ Fixed Point and Stochastic Integrability

It is the unique time homogenous Markov process $(\mathfrak{h}_t(\cdot;\mathfrak{h}), t \geq 0)$ taking place on UC (upper semicontinuous functions plus growth control) with transition probabilities on cylindrical sets given by

$$\mathbb{P}^{\mathfrak{h}}\left(\cap_{i=1}^{m} \{\mathfrak{h}_t(x_i) \leq y_i\} \right) = \det(I - K)_{L^2(\{x_1,\dots,x_m\}\times\mathbb{R})}, \quad (1)$$

where $K = K(\mathfrak{h}, y, t)$ is the Brownian Scattering operator as introduced by Matetski, Quastel and Remenik (2017). The time evolution of the transition probabilities can be linearized through K (stochastic integrability).

Examples

Initial Profiles

- Narrow Wedge at $x \in \mathbb{R}$: $\mathfrak{h} \equiv \mathfrak{d}_x$ where

$$\mathfrak{d}_x(z) = \begin{cases} 0 & \text{for } z = x \\ -\infty & \text{for } z \neq x \,. \end{cases}$$

- Flat: $\mathfrak{h} \equiv 0$.

- Stationary: $\mathfrak{h} \equiv \mathfrak{b}$ a two-sided BM with $\sigma = 2$.

Remark

The initial profile of particles $h^{(n)}$ might depende on n, in such way that for any $\mathfrak{h} \in UC$ one can build a sequence of initial particle profiles $h^{(n)}$ such that $\mathfrak{h}_{n,0} \to \mathfrak{h}$.

Symmetries

The "scaling" ($\gamma > 0$) and "vertical shift" operators acting on real functions $\mathfrak{f}$ are denoted as

$$S_\gamma \mathfrak{f}(x) := \gamma^{-1} \mathfrak{f}(\gamma^2 x) \quad \text{and} \quad \Delta \mathfrak{f}(x) := \mathfrak{f}(x) - \mathfrak{f}(0)\,,$$

respectively.

- 1-2-3 Scaling: $S_{\gamma^{-1}} \mathfrak{h}_{\gamma^{-3} t}(\cdot\,; S_\gamma \mathfrak{h}) \overset{dist.}{=} \mathfrak{h}_t(\cdot\,; \mathfrak{h})$. In particular, for $\gamma_t := t^{1/3}$,

$$\mathfrak{h}_t(\cdot\,; \mathfrak{h}) \overset{dist.}{=} S_{\gamma_t^{-1}} \mathfrak{h}_1(\cdot\,; S_{\gamma_t} \mathfrak{h})\,, \quad \text{for all} \quad t > 0\,.$$

- Time Stationarity: let $\mathfrak{b}^\mu(x) := \mu x + \mathfrak{b}(x)$. Then

$$\Delta \mathfrak{h}_t(\cdot\,; \mathfrak{b}^\mu) \overset{dist.}{=} \mathfrak{b}^\mu(\cdot)\,, \quad \text{for all} \quad t \geq 0\,.$$

Long Time Behaviour of the KPZ Fixed Point

Ergodicity

- Find a sufficient and necessary condition on the initial profile $\mathfrak{h}$ such that

$$\lim_{t\to\infty} \Delta\mathfrak{h}(\cdot;\mathfrak{h}) \overset{dist.}{=} \mathfrak{b}(\cdot)\,.$$

- Is $\{\mathfrak{b}^{\mu} \,:\, \mu \in \mathbb{R}\}$ the only collection of time stationary and spatially ergodic (in terms of increments) processes for the KPZ fixed point?

Long Time Behaviour of the KPZ Fixed Point

Stochastic Integrability

The description of the transition probabilities in terms of Fredholm determinants (1) is suitable to prove finite dimensional convergence to $\mathfrak{h}$ for suitable initial conditions. Matetski, Quastel and Remenik (2017)

Coupling Method

An alternative description of the KPZ fixed point using the directed landscape constructed by Dauvergne, Ortmann and Virag (2018) allow us to use particle systems techniques, such as attractiveness and comparison (under a basic coupling), which provide stronger results making use of a simpler approach.

The Airy Sheet

Dauvergne, Ortmann and Virag (2018) showed the existence of a translation invariant and symmetric two-dimensional scalar field, called the Airy Sheet, such that

$$\mathcal{A}(x, y) = \mathfrak{h}_1(y; \mathfrak{d}_x) + (y - x)^2 .$$

Furthermore, for fixed $y \in \mathbb{R}$, $\{\mathcal{A}(x, y) \,:\, x \in \mathbb{R}\}$ is distributed as the Airy$_2$ process.

The Directed Landscape

There exists a unique space-time continuous random scalar field,

$$\left\{\mathcal{L}(z,s;x,t);\, s,t \in \mathbb{R} \text{ with } s<t\,,\ (x,y)\in\mathbb{R}^2\right\}\,,$$

called the directed landscape. It enjoys a metric composition:

$$\mathcal{L}(x,r;y,t) = \max_{z\in\mathbb{R}}\left\{\mathcal{L}(x,r;z,s)+\mathcal{L}(z,s;y,t)\right\}\,. \tag{2}$$

The Directed Landscape

It also satisfies the following symmetries (as two-dimensional continuous processes):

$$\mathcal{L}(z,0;x,t) \stackrel{dist.}{=} S_{\gamma_t^{-1}}\mathcal{A}(z,x) - \frac{(x-z)^2}{t}\,,$$

and

$$\mathcal{L}(z,s;x,t+s) \stackrel{dist.}{=} \mathcal{L}(z,0;x,t)\,.$$

Furthermore, for $r < s \leq t < u$ fixed $\mathcal{L}(z,r;x,s)$ is independent of $\mathcal{L}(z,t;x,u)$.

The KPZ Fixed Point and The Directed Landscape

The space-time process defined as

$$\mathfrak{h}_{s,t}(x;\mathfrak{h}) := \max_{z\in\mathbb{R}} \left\{\mathfrak{h}(z) + \mathcal{L}(z,s;x,t)\right\} , \tag{3}$$

for $s < t$, is distributed as the KPZ fixed point at time t, starting at $\mathfrak{h}$ at time s, so that $\mathfrak{h}_t \equiv \mathfrak{h}_{0,t}$.

Basic Coupling

Given $\mathfrak{h}_1 \mathfrak{h}_2 \in UC$, consider the coupling $(\mathfrak{h}_t(\cdot;\mathfrak{h}_1), \mathfrak{h}_t(\cdot;\mathfrak{h}_2))$, constructed from (3):

$$\mathfrak{h}_{s,t}(x;\mathfrak{h}) = \max_{z\in\mathbb{R}} \left\{\mathfrak{h}(z) + \mathcal{L}(z,s;x,t)\right\} ,$$

and

$$\mathfrak{h}_{s,t}(x;\mathfrak{h}) = \max_{z\in\mathbb{R}} \left\{\mathfrak{h}(z) + \mathcal{L}(z,s;x,t)\right\} .$$

Theorem

Let $\gamma > 0$ and assume that there exist $c > 0$ and a real function ψ, that does not depend on $\gamma > 0$, such that $\lim_{r\to\infty} \psi(r) = 0$ and for all $\gamma \geq c$ and $r \geq 1$

$$\mathbb{P}\left(S_\gamma \mathfrak{h}(z) \leq r|z| \, , \forall \, |z| \geq 1 \right) \geq 1 - \psi(r) \, . \tag{4}$$

Let $a, t, \eta > 0$ and set $r_t := \sqrt[4]{t^{2/3} a^{-1}}$. Under the coupling (3), where $\mathfrak{b}$ and $\mathfrak{h}$ are sample independently, there exists a real function ϕ, which does not depend on $a, t, \eta > 0$, such that $\lim_{r\to\infty} \phi(r) = 0$ and for all $t \geq \max\{c^3, a^{3/2}\}$ and $\eta > 0$ we have

$$\mathbb{P}\left(\sup_{x\in[-a,a]} |\Delta\mathfrak{h}_t(x;\mathfrak{h}) - \Delta\mathfrak{h}_t(x;\mathfrak{b})| > \eta\sqrt{a} \right) \leq \phi\left(r_t\right) + \frac{1}{\eta r_t} \, .$$

Proof

For the proof we use the metric composition (2) to prove attractiveness and comparison under coupling (3). This allows us to show that if a certain event $E_t(a)$ occurs, then

$$\sup_{x \in [-a,a]} |\Delta \mathfrak{h}_t(x;\mathfrak{h}) - \Delta \mathfrak{h}_t(x;\mathfrak{b})| \leq I_t(a)\,,$$

where $I_t(a)$ is a nonnegative random variable such that

$$\mathbb{E} I_t(a) \leq \frac{\sqrt{a}}{r_t}\,.$$

Using the symmetries of $\mathcal{L}$, we can show that under assumption (4)

$$\mathbb{P}\left(E_t(a)\right) \leq \phi(r_t)\,.$$