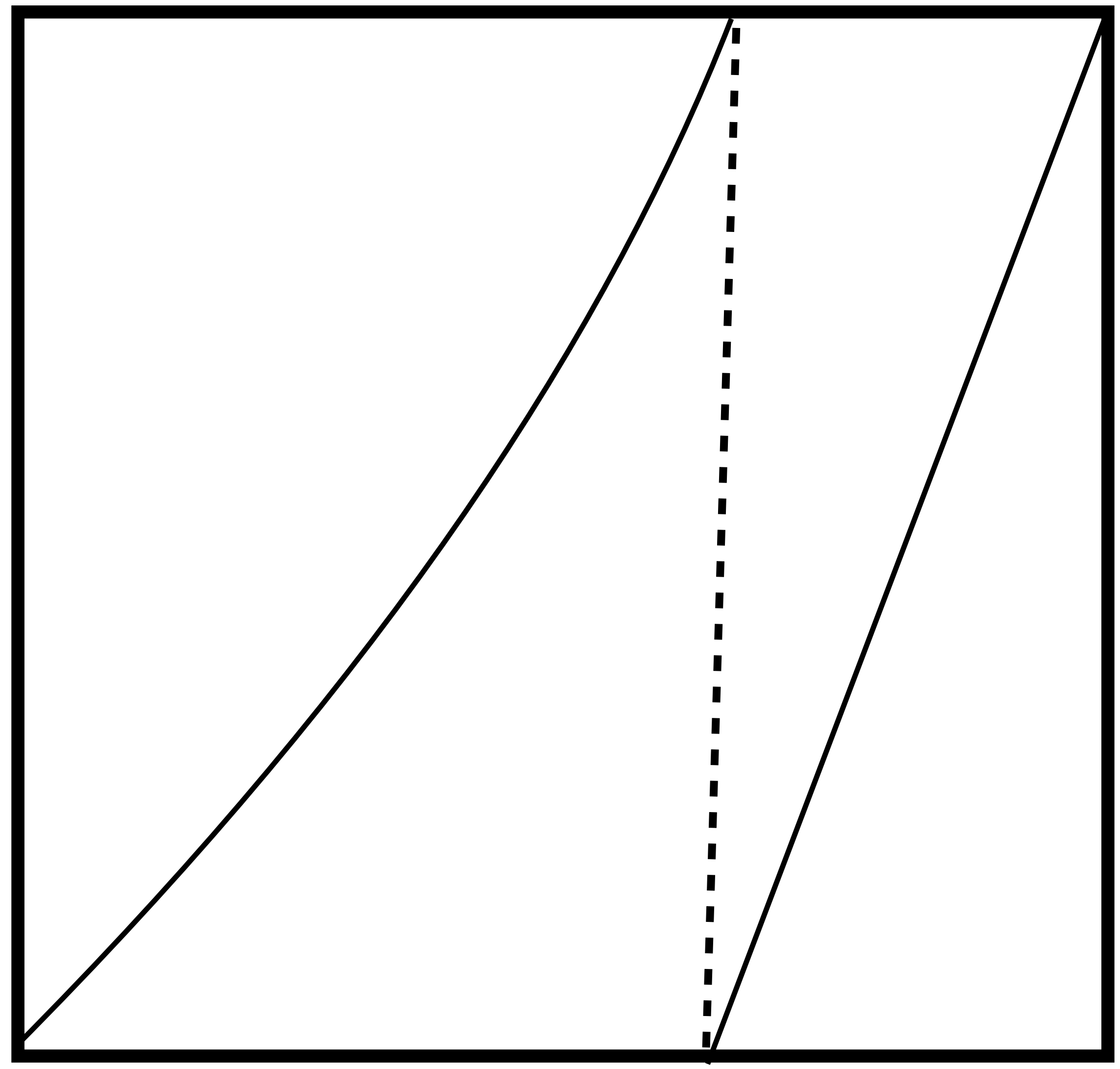
Transfer operators and Besov spaces

Daniel Smania
ICMC-USP-Brazil

joint work with Alexander Arbieto (UFRJ-Brazil)

Introduction

C^2 Expanding Markovian maps



F

F has an unique invariant probability that is absolutely continuous with respect to Lebesgue measure.

Randon variables

$$\phi\colon I \to \mathbb{R} \text{ is a given observable}$$

Consider the random variables

$$\phi, \phi \circ F, \phi \circ F^2, \ldots, \phi \circ F^n, \ldots$$

-Those are identically distributed with respect to the invariant probability

Ergodic Theory

Study of the statistical properties of this sequence of random variables

The commander-in-chief

Exponential decay of correlations

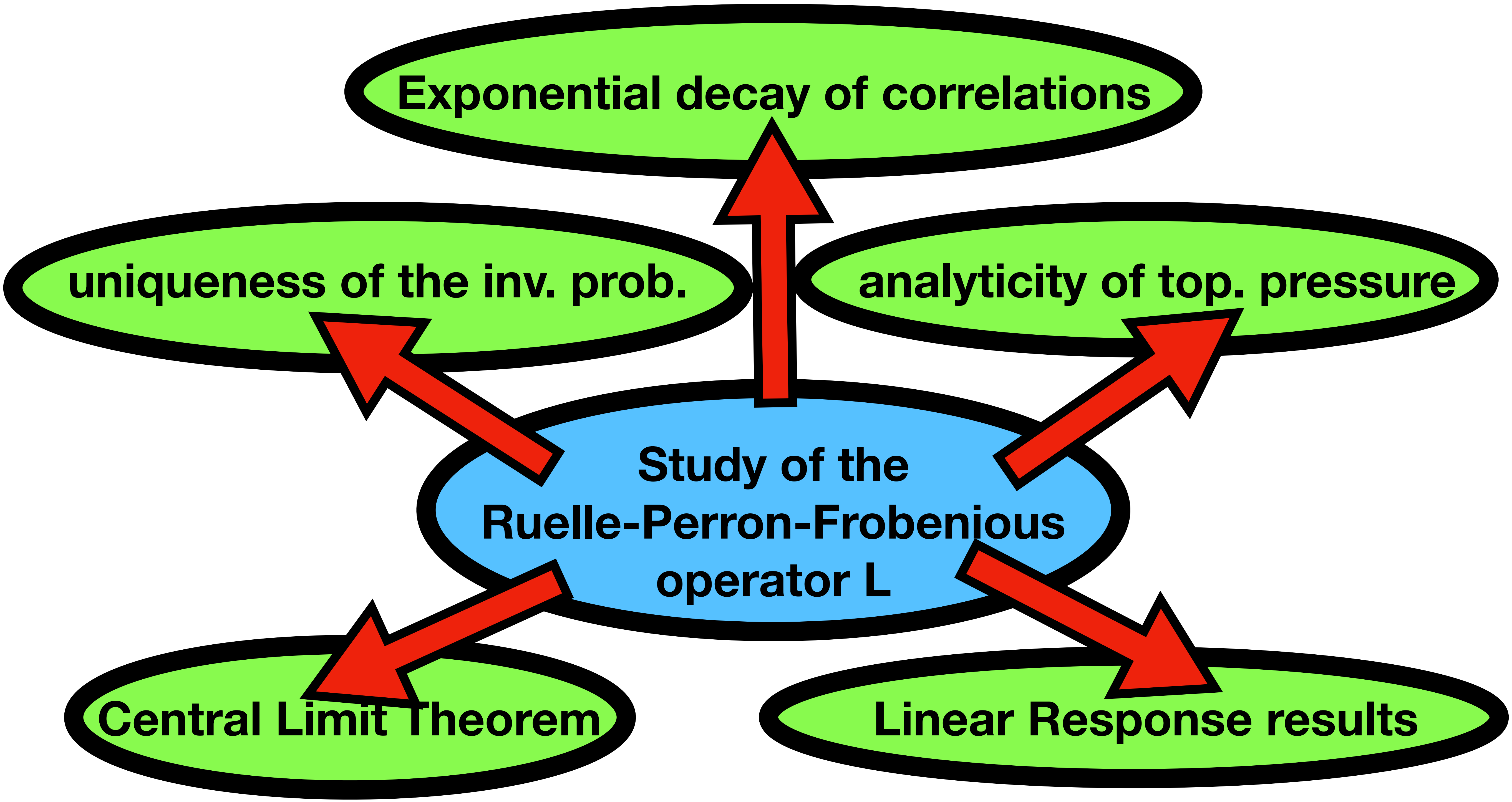
uniqueness of the inv. prob.

analyticity of top. pressure

Central Limit Theorem

Linear Response results

The commander-in-chief

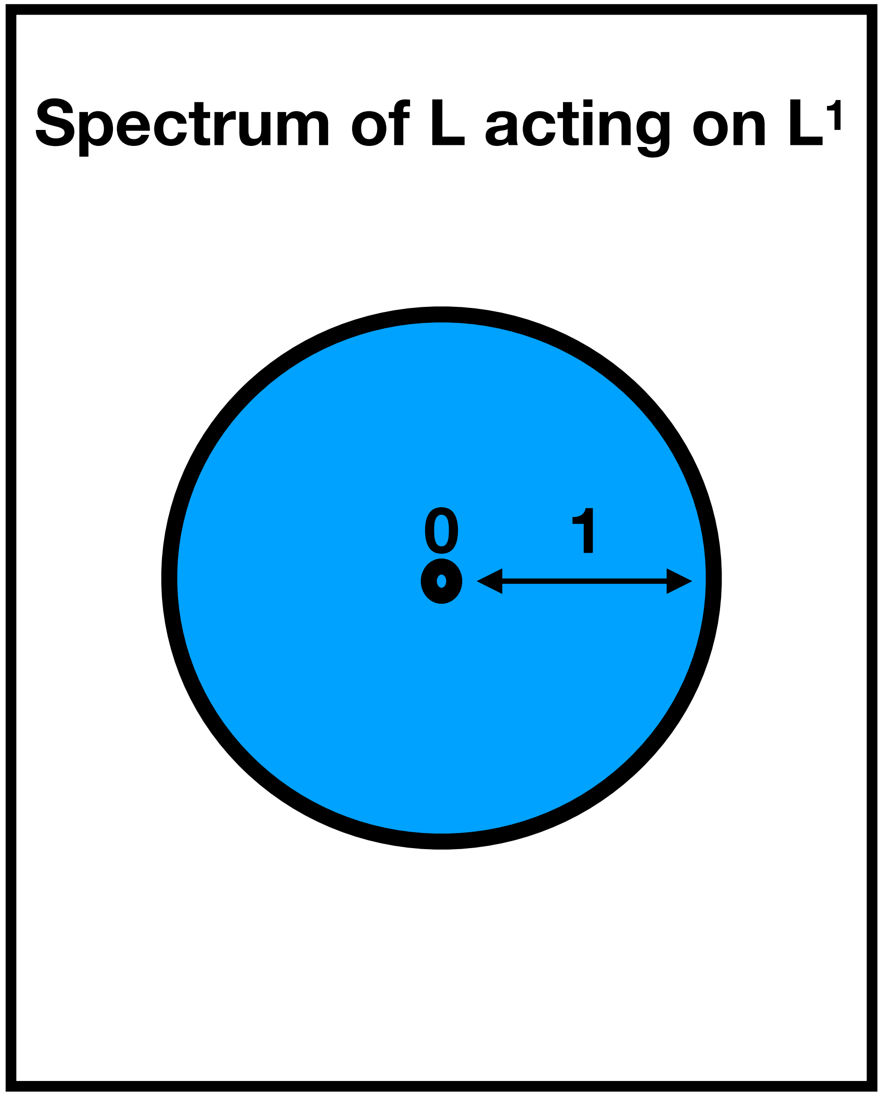


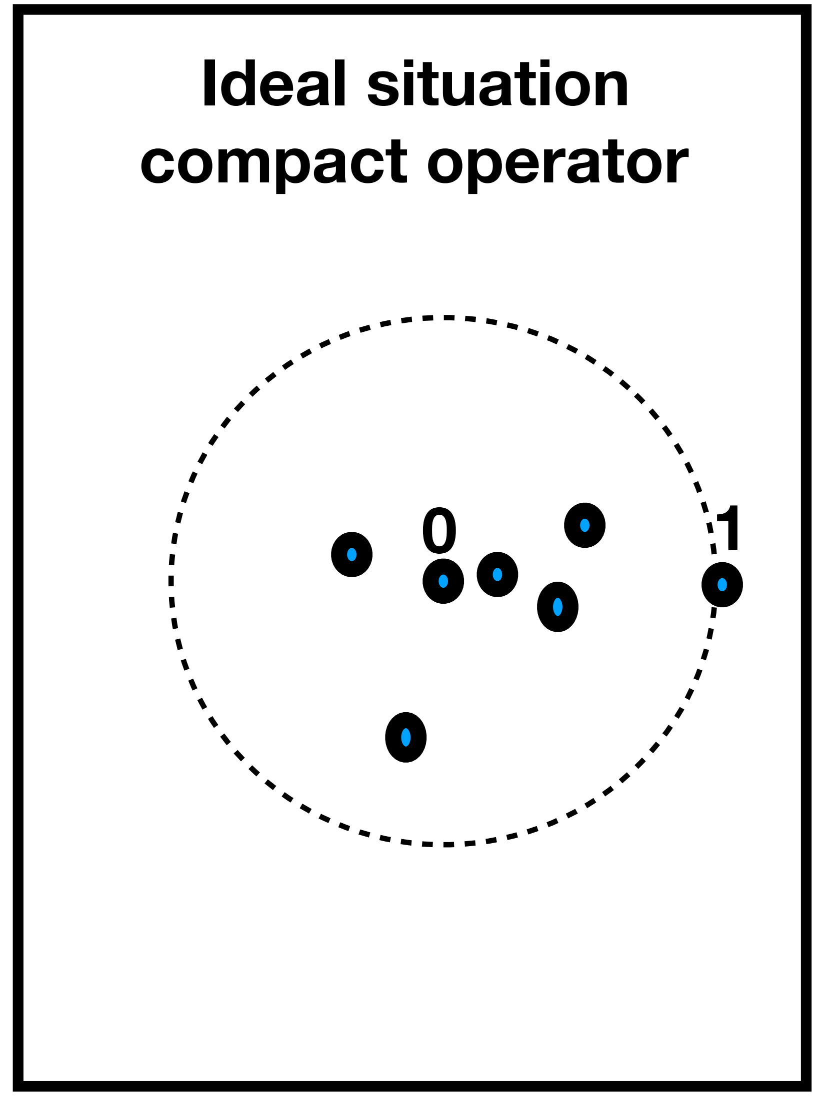
Ruelle-Perron-Frobenious operator L

$$L\colon L^1(m) \to L^1(m)$$

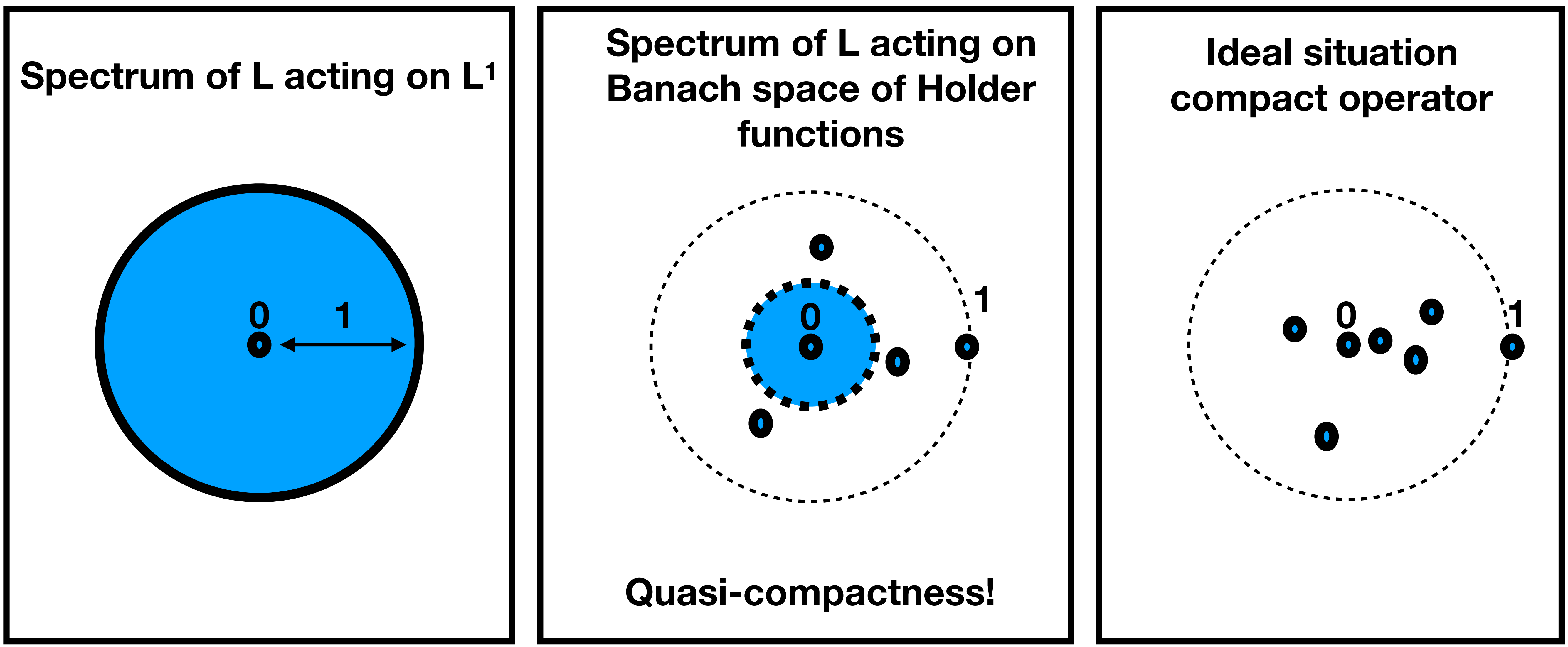
$$(L\phi)(x) = \sum_{F(y)=x} |DF(y)|^{-1}\phi(y)$$

Quasi-compactness: Ruelle-Perron-Frobenious operator L





Quasi-compactness: Ruelle-Perron-Frobenious operator L



Ongoing quest in metric theory of expanding maps:

With weaker assumptions on the expanding map we obtain a wide class of observables for which exponential decay of correlations, central limit theorem, etc… holds.

Examples:

Piecewise C^2 ($C^{1+alpha}$) expanding maps (non-markovian), Lorenz 1-d maps (derivative goes to infinity at some points), 1-d expanding maps with infinitely many branches, piecewise multidimensional measure-expanding maps, expanding maps defined on fractals.

A bit of history

Markovian expanding maps: Gauss, Borel, Rényi, Ulam, Krzyżewski&Szlenk, Ruelle, Sinai, Walters, Parry, Pollicott.

Lasota&Yorke, Keller&Hofbauer: Piecewise C^2 1-d expanding maps (estimate-free)

BV space

Góra and A. Boyarsky: Piecewise C^2 expanding maps

BV space

Keller: Piecewise C^{1+alpha} 1-d expanding maps (estimate-free)

generalised p-bounded variation space.

Saussol: piecewise multidimensional expanding maps.

generalised p-bounded variation space.

Tsujii, Buzzi: piecewise real analytic multidimensional expanding maps (estimate-free).

generalised p-bounded variation space.

Recent stuff

Tsujii, Avila-Gouezel-Tsujii (Fat Solenoidal attractor. Smooth maps)

Sobolev space

Thomine (piecewise smooth maps on surfaces, inspired in Baladi-Gouezel)

Sobolev space

Liverani (piecewise C^{1+\alpha} with an infinity number of branches and Holder potentials)

Space of certain regular measures.

Butterley (piecewise C^{1+\alpha} with finite many branches and Holder potentials)

Space of certain regular measures.

Eslami (Expanding maps on metric spaces)

Standard pairs (Chernov-Dolgopyat)

Nakano&Sakamoto (smooth expanding maps on manifolds)

Besov spaces

We want to give yet another Banach space of functions where quasi-compactness of the RPF-operator holds, even when the map and/or phase space and/or potential are very irregular.

WHY?????????

Selling points

1. The Banach space has a very elementary definition.

2. In lots of situations this Banach space includes previous Banach spaces in literature (in particular the Keller space of generalised p-bounded variation)

3. In many situations these Banach space coincides with Besov spaces!

4. We have results for Besov potentials! (but it is not a estimate-free result)

5. The reason for the quasi-compactness of L is very easy to explain.

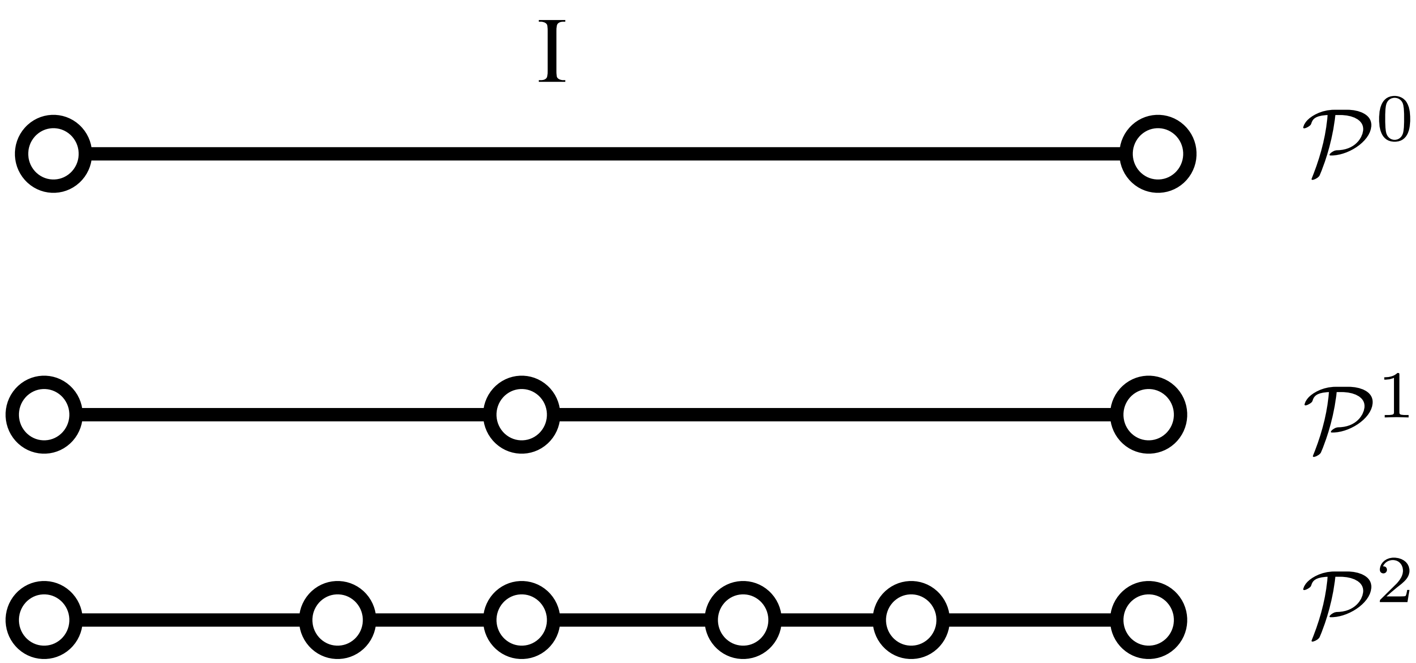
6. Unified approach for Lorenz maps, maps defined in fractals and multidimensional piecewise smooth expanding maps with fractal domains.

Our
Banach Space(s)

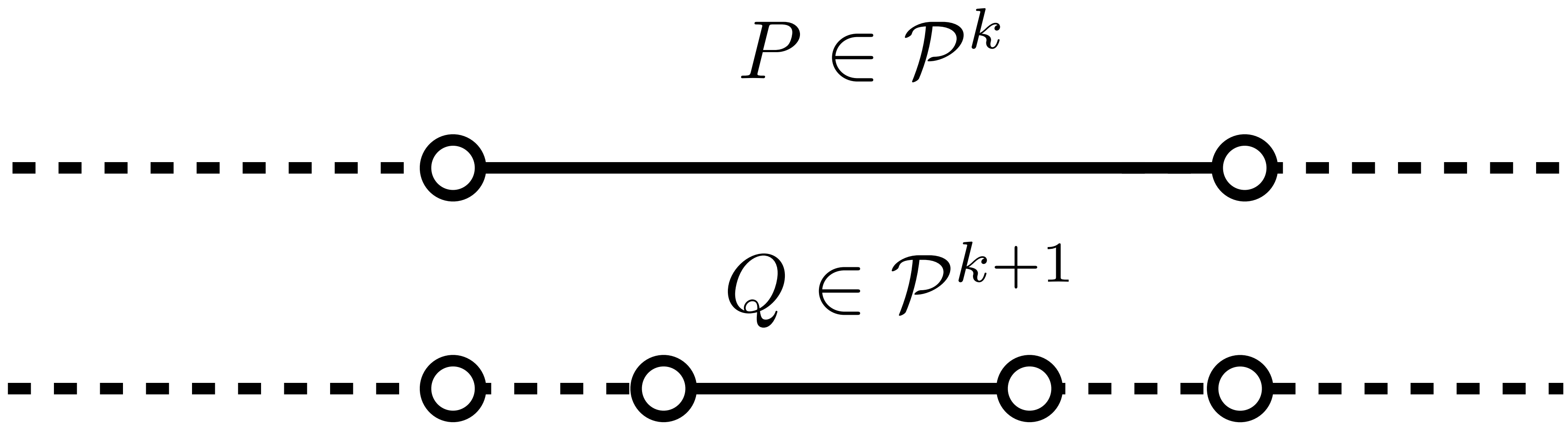
Setting

(I, m) measure space with probability measure m.

$\mathcal{P}^k$ sequence of nested partitions of I with bounded geometry.



Bounded Geometry

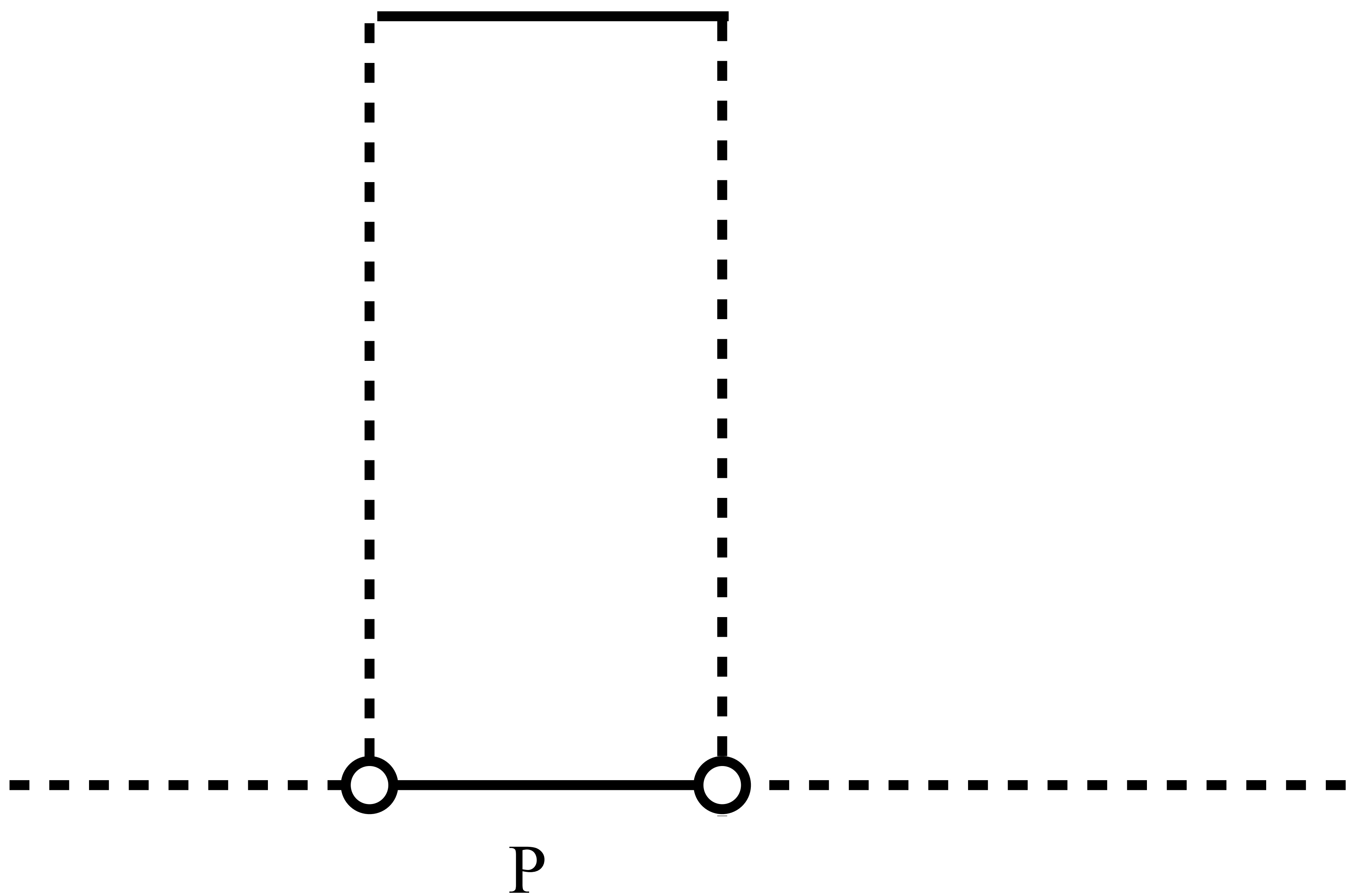


$$0 < \lambda_1 \leq \frac{|Q|}{|P|} \leq \lambda_2 < 1.$$

(s,p)-Souza's Atoms.

$P \in \mathcal{P}^k$

$$a_P(x) = \begin{cases} |P|^{s-1/p}, \; if \; x \in P, \\ 0, \; if \; x \notin P. \end{cases}$$



Besov space $\mathcal{B}^s_{p,q} = \mathcal{B}^s_{p,q}(I, m, (\mathcal{P}^k)_k)$

$$0 < s < 1/p,\ p \geq 1,\ q \geq 1$$

Atomic Decomposition

$$\phi = \sum_k \sum_{P \in \mathcal{P}^k} c_P a_P$$

$$\Big(\sum_k \Big(\sum_{P \in \mathcal{P}^k} |c_P|^p\Big)^{q/p}\Big)^{1/q} < \infty$$

$$0 < s < 1/p,\ p \geq 1,\ q \geq 1$$

Atomic Decomposition

Suppose that

$$\phi = \sum_k \sum_{P \in \mathcal{P}^k} c_P a_P$$

converges absolutely on L^p. The (s,p,q)-cost of this representation is

$$\Big(\sum_k \Big(\sum_{P \in \mathcal{P}^k} |c_P|^p \Big)^{q/p} \Big)^{1/q}$$

Besov space $\mathcal{B}^s_{p,q} = \mathcal{B}^s_{p,q}(I, m, (\mathcal{P}^k)_k)$

$\phi \in L^p(m)$ belongs to $\mathcal{B}^s_{p,q}$ iff ϕ has a presentation with finite (s,p,q)-cost.

arXiv.org > math > arXiv:1903.06943

Mathematics > Dynamical Systems

Transfer operators and atomic decomposition

Alexander Arbieto, Daniel Smania

arXiv.org > math > arXiv:1903.06976

Mathematics > Dynamical Systems

Transfer operators, atomic decomposition and the Bestiary

Daniel Smania

History

Coifman (1974): Atomic decomposition of real Hardy spaces of R

Frazier and B. Jawerth (1985): Atomic decomposition (smooth atoms) of Besov spaces in R^n.

G. de Souza (1985): Atomic decomposition (Souza's atoms) of certain Besov spaces in R.

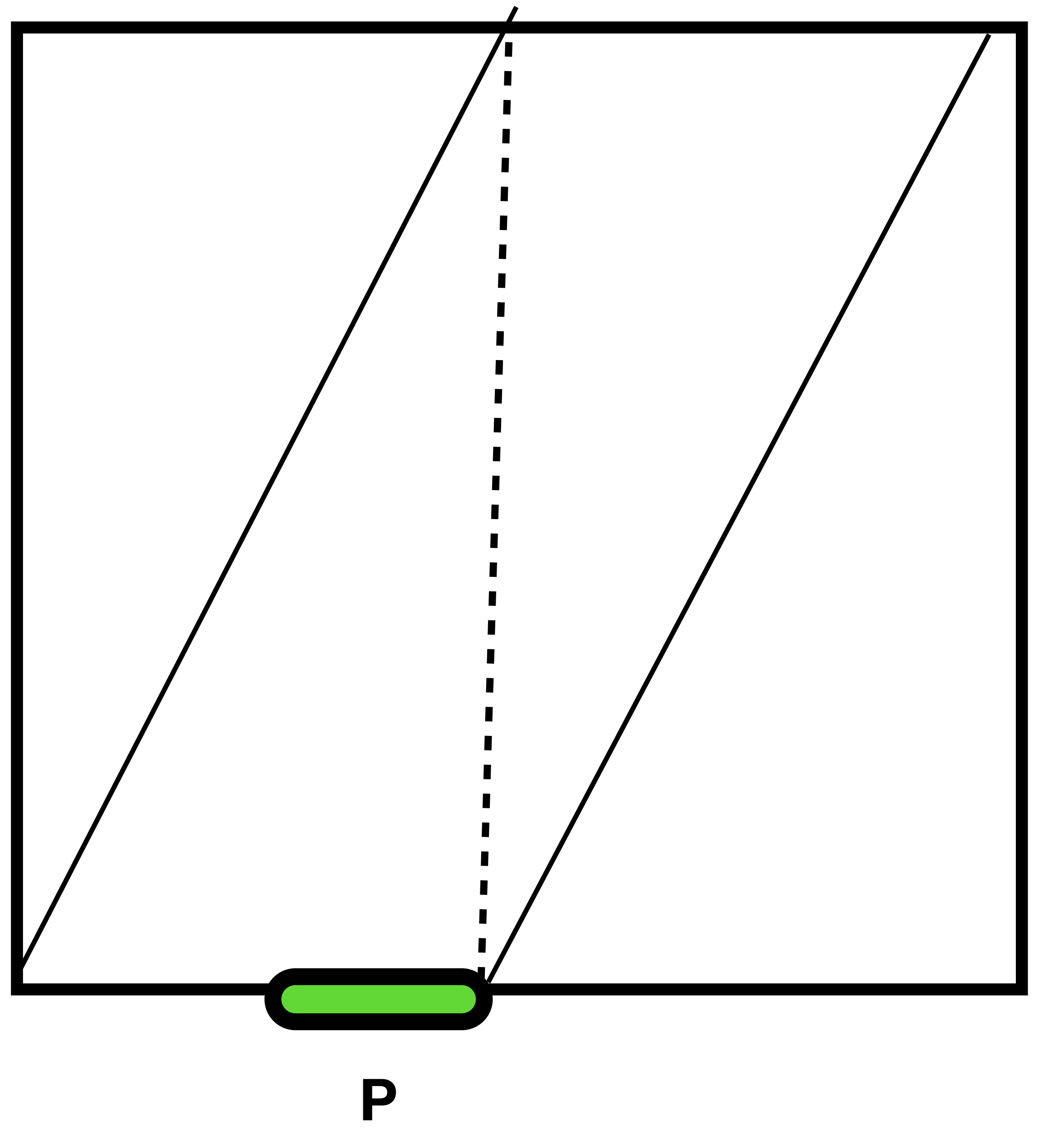
Gu and Taibleson (1992): Martingale Besov spaces.

Han and Sawyer (1994) and Han, Lu and Yang (1999): Atomic

decomposition (with Holder atoms) of homogeneous spaces.

DeVore and V. A. Popov (1988): atomic decomposition (splines) for Besov spaces of [0,1]^n.

Why we have quasi-compactness of L?



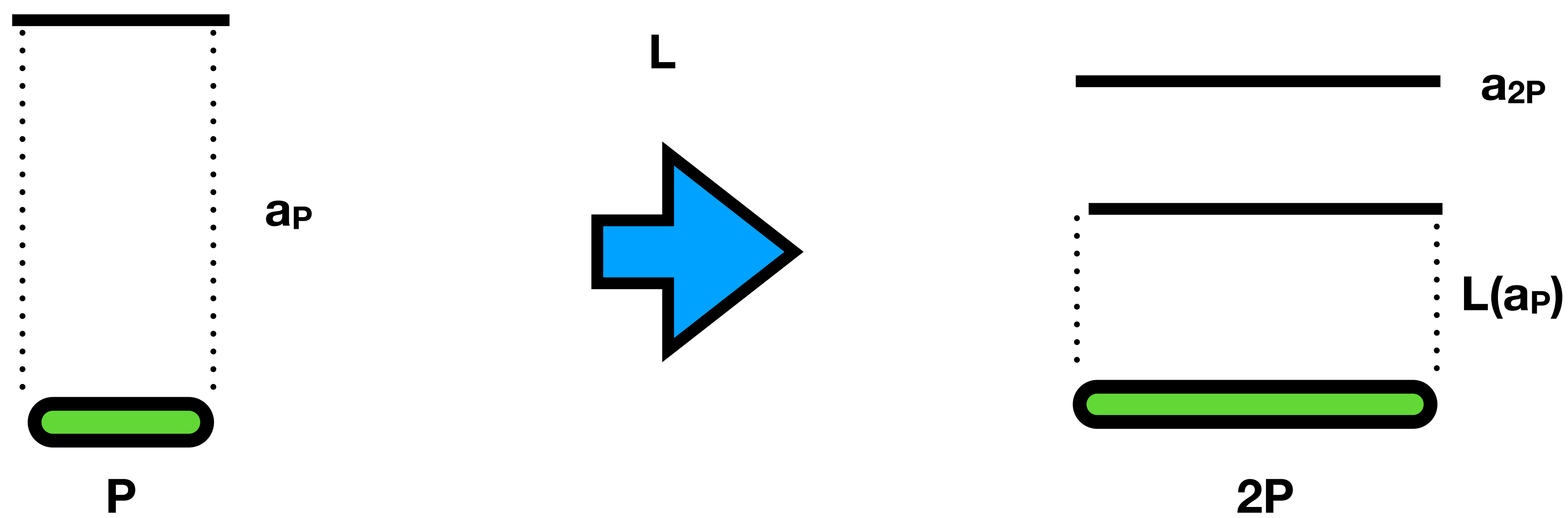
F(x)=2x mod 1

$$L(a_P) = L(|P|^{s-1/p} 1_P) = \frac{|P|^{s-1/p}}{2} 1_{2P}$$

$$= \frac{1}{2^{1-(1/p-s)}} |2P|^{s-1/p} 1_{2P} = \frac{1}{2^{1-(1/p-s)}} a_{2P}$$

< 1

For small atoms (support smaller than 1/2), the image by L of an atom is a small fraction of an atom.



$$\phi = \sum_{k} \sum_{P \in \mathcal{P}^k} c_P a_P$$

small atoms

large atoms (finite number of them!)

$$= \sum_{k > k_0} \sum_{P \in \mathcal{P}^k} c_P a_P + \sum_{k \leq k_0} \sum_{P \in \mathcal{P}^k} c_P a_P .$$

L contracts this part!

Finite rank part!

L= contraction + Finite rank -> quasi-compactness

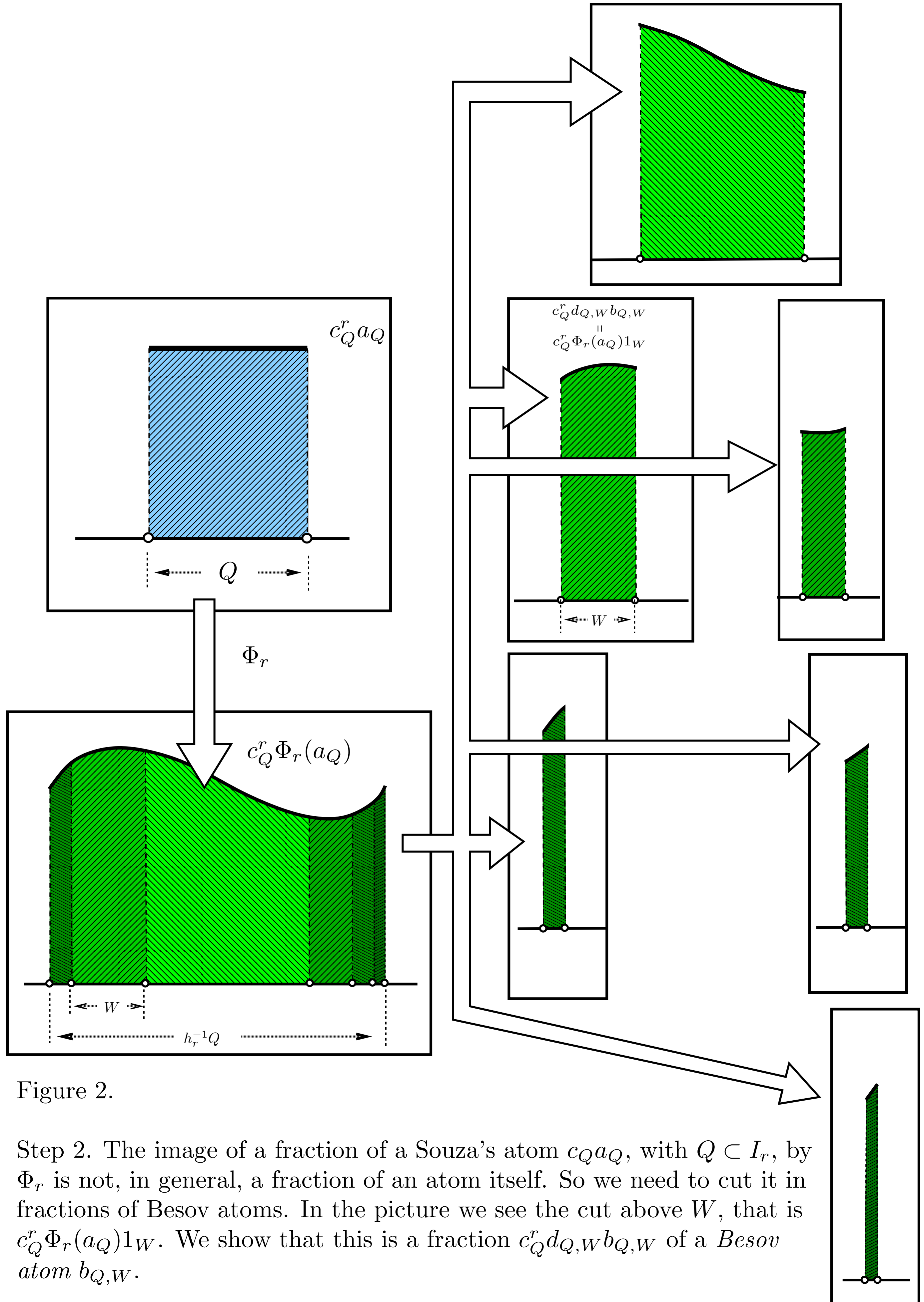


Figure 2.

Step 2. The image of a fraction of a Souza's atom $c_Q a_Q$, with $Q \subset I_r$, by Φ_r is not, in general, a fraction of an atom itself. So we need to cut it in fractions of Besov atoms. In the picture we see the cut above W, that is $c_Q^r \Phi_r(a_Q) 1_W$. We show that this is a fraction $c_Q^r d_{Q,W} b_{Q,W}$ of a *Besov atom* $b_{Q,W}$.