# Local Convergence of Spatial Gibbs Random Graphs

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# Spatial Gibbs Measures

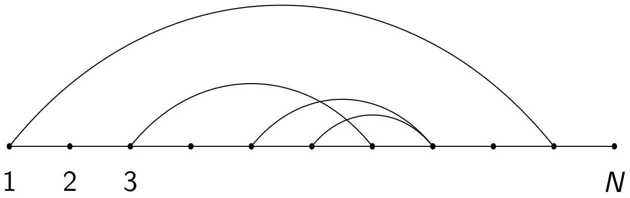
- Introduced by J.-C Mourrat and D. Valesin.

$$\mathcal{G} = \{g = (V, E) : \; V \subseteq \mathbb{Z} \text{ and } g \text{ is locally finite}\}.$$

- $N \geq 1$ natural number. Define $g = (V_N, E_N) \in \mathcal{G}$, where

$$V_N = \{1, \dots, N\},$$
$$E_N \supseteq \{\{i, i+1\} : \; 1 \leq i < N\}.$$

- $\mathcal{G}_N$ be the set of these graphs.



- Consider $p \in [1, \infty]$.
- For $g \in \mathcal{G}_N$, define, for $p \in [1, \infty)$,

$$\mathcal{H}_p(g) = \begin{cases} \left( \frac{1}{\binom{N}{2}} \sum_{\substack{x,y \in V_N: \\ x<y}} (\mathrm{d}_g(x,y))^p \right)^{\frac{1}{p}} & \text{if } p \in [1,\infty); \\ \sup\{\mathrm{d}_g(x,y) : \; x,y \in V_N\} & \text{if } p = \infty, \end{cases}$$

- $\mathrm{d}_g(x, y)$ is the graph-theoretic distance of the graph $g$.
- $\mathcal{H}(g) := \mathcal{H}_\infty(g)$ is the diameter of the graph $g$.

- Consider $\gamma > 0$.

$\mathbb{P}_{N,\gamma}$ on $\mathcal{G}_N$: independently, each edge $xy$, $|x-y|>1$, is present in the graph with probability $p_{xy} = \exp(-|x-y|^\gamma)$.

Question: What is the typical value of $\mathcal{H}(g)$ under $\mathbb{P}_{N,\gamma}$?

Answer: $\mathcal{H}(g) = O(N)$.

**Problem**

*Find a new probability measure where the typical graphs have diameter smaller than $O(N)$.*

Let $b \in \mathbb{R}$.

$$\mathbb{P}^b_{N,\gamma}(g) = \frac{1}{Z^b_{N,\gamma}} e^{-N^b \mathcal{H}(g)} \cdot \mathbb{P}_{N,\gamma}(g),$$

where $Z^b_{N,\gamma}$ is the partition function.

- $e^{-N^b \mathcal{H}(g)}$ avoids large diameters.
- $\mathbb{P}_{N,\gamma}(g)$ avoids long edges.

**Theorem (Mourrat, Valesin - Ann. Appl. Probab. '18)**

*For every $\alpha \in [0,1]$, there exist $\gamma > 0$ and $b \in \mathbb{R}$ such that, for every $\varepsilon > 0$,*

$$\lim_{N\to\infty} \mathbb{P}^b_{N,\gamma}\left( \left| \frac{\log \mathcal{H}(G)}{\log N} - \alpha \right| < \varepsilon \right) = 1.$$

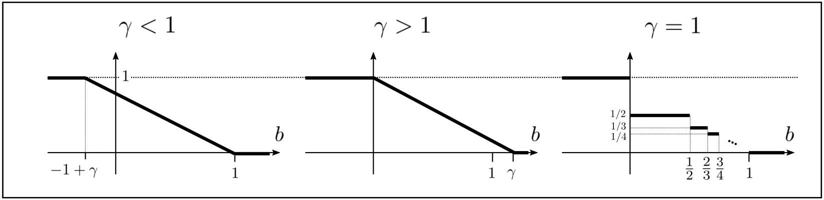



Figure: Plot of the function $b \mapsto \alpha(\gamma, b)$ for the three cases $\gamma \in (0,1)$, $\gamma > 1$ and $\gamma = 1$.

Define $\mathcal{E} = [0, \frac{1}{4}] \cup \bigcup_{k=2}^{\infty} \left\{ \frac{k-1}{k} \right\}$ the missing points when $\gamma = 1$.

# Main Problem

Define $\mathbb{P}_\gamma$ be the probability measure on

$$\mathcal{G}_{\mathbb{Z}} = \{g \in \mathcal{G} : V(g) = \mathbb{Z},\ E(g) \supset \{xy : |x-y| = 1\}\}.$$

so that each edge $xy$, $|x-y| > 1$, are independent with probability

$$p_{xy} = \exp\left(-|x-y|^\gamma\right).$$

Problem: Does $\mathbb{P}^b_{N,\gamma}$ converge when $N \to \infty$?
Convergence: In the sense of Benjamini-Schramm...stronger version.

- $\mathcal{G}_\bullet = \{(g, o) : g \in \mathcal{G}, o \text{ is a vertex of } g\}$.
- Topology: Given $R > 0$, $(g, o) \in \mathcal{G}_\bullet$,

$$\begin{aligned} B_{(g,o)}(R) &= ((V_B, E_B), o) \in \mathcal{G}_\bullet, \\ V_B &= \{x \in V : \mathrm{d}_g(o, x) \leq R\}, \\ E_B &= \{\{x, y\} \in E : \mathrm{d}_g(o, x) \leq R \text{ and } \mathrm{d}_g(o, y) \leq R\}. \end{aligned}$$

- Isomorphism: For a fixed $o, o' \in \mathbb{Z}$, define $\varphi_{o,o'} : \mathbb{Z} \to \mathbb{Z}$

$$\varphi_{o,o'}(x) = x - o + o'.$$

- Convergence of graphs: $(g_n, o_n) \to (g, o)$ if

$$\forall R > 0, \exists n_0 \geq 1, \forall n \geq n_0,\ \varphi_{o,o'}(B_{(g_n,o_n)}(R)) = B_{(g,o)}(R).$$

- Convergence in distribution: $\mu_n \to \mu$ if $\forall R > 0, \forall (g, o) \in \mathcal{G}_\bullet$,

$$\lim_{n\to\infty} \mu_n(B_{(G_n,\mathcal{O}_n)}(R) \simeq B_{(g,o)}(R)) = \mu(B_{(G,\mathcal{O})} \simeq B_{(g,o)}(R)).$$

**Theorem (Endo, Valesin - 2017)**

*Assume that either of the following conditions hold:*

$$[\gamma \in (0,1),\ b \in (-\infty, 1)],\ [\gamma = 1,\ b \in (-\infty, 1)\backslash\mathcal{E}],$$
$$[\gamma > 1,\ b \in (-\infty, 0)].$$

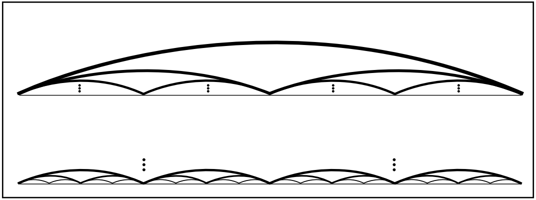
*Let* $\mathcal{U}_N$ *be the uniform measure on* $\{1, \ldots, N\}$*. Then*

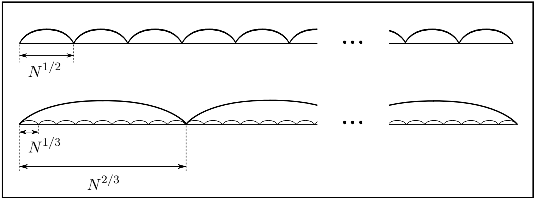
$$\mathbb{P}^b_{N,\gamma} \otimes \mathcal{U}_N \to \mathbb{P}_\gamma \otimes \delta_{\{0\}}.$$

# Heuristic

Existence of a graph $g^\star = g^\star(N, \gamma, \alpha)$ with $\mathcal{H}_p(g^\star)$ close to $N^\alpha$.

$$\mathbb{P}_{N,\gamma}(\mathcal{H}_p(G_N) \leq N^\alpha) \geq \mathbb{P}_{N,\gamma}(g^\star \text{ is a subgraph of } G_N).$$





# Main tools for the proof

Under conditions on $\gamma$, $b$, it is enough to show $\forall \varepsilon > 0, \forall (g,o) \in \mathcal{G}_\bullet$,

$$\mathbb{P}^b_{N,\gamma}\left(\left|\frac{\#\{i \in [N] : B_{(G_N,i)}(k) \simeq (g,o)\}}{N} - \mu_\gamma(k,(g,o))\right| > \varepsilon\right) \xrightarrow{N\to\infty} 0.$$

## Lemma

1. *Under conditions on $\gamma$, $b$: If $E_N$ are events with $\mathbb{P}_{N,\gamma}(E_N) < \exp\{-\beta N\}$ for some $\beta > 0$ and $N$ large, then $\mathbb{P}^b_{N,\gamma}(E_N) \xrightarrow{N\to\infty} 0$.*
2. *Assume $\gamma = 1$ and $b \in [0,1)\backslash\mathcal{E}$. Then,*
   - 2a. *there exists $C > 0$ such that, if $E_N$ are events with $\mathbb{P}_{N,1}(E_N) < \exp\{-CN\}$ for all $N$, then $\mathbb{P}^b_{N,1}(E_N) \xrightarrow{N\to\infty} 0$;*
   - 2b. *if $E_N$ are events such that $\mathbb{P}_{N,1}(E_N) < \exp\{-cN\}$ for some $c > 0$, and each $E_N$ only depends on $\{e : |e| \le L\}$ for a fixed $L$, then $\mathbb{P}^b_{N,1}(E_N) \xrightarrow{N\to\infty} 0$.*

# Quasi-independent random variables

We need a concentration result for sums of bounded random variables with finite-range dependence.

**Lemma (Janson - Random Struct., '04)**

*Let $Y_1, \ldots, Y_n$ be random variables such that, for some $m, L > 0$ and for each $i$, $0 \leq Y_i \leq m$ and $Y_i$ is independent of $\{Y_j : |j - i| > L\}$. Then, letting $X = \sum_{i=1}^{n} Y_i$, we have*

$$\mathbb{P}\left(|X - \mathbb{E}(X)| > t\right) \leq 2 \exp\left\{-\frac{2t^2}{(2L+1)nm^2}\right\}.$$

# References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