Directed polymer in γ-stable Random Environments

Roberto Viveros

IMPA. Rio de Janeiro-Brazil

JULY 23, 2019

Chemical background

Chemical background

A *polymer* is a large molecule consisting of monomers that are tied together by chemical bonds.

Chemical background

A *polymer* is a large molecule consisting of monomers that are tied together by chemical bonds.

The monomers can be either small units, such as CH2 in polyethylene; or larger units with an internal structure (such as the adenine-thymine and cytosine-guanine base pairs in the DNA double helix.

Chemical background

A *polymer* is a large molecule consisting of monomers that are tied together by chemical bonds.

The monomers can be either small units, such as CH2 in polyethylene; or larger units with an internal structure (such as the adenine-thymine and cytosine-guanine base pairs in the DNA double helix.

Polymers abound in nature because of the multivalency of atoms like carbon, silicon, oxygen, nitrogen, sulfur and phosphorus, which are capable of forming long concatenated structures.

Polymer Classification.

- Homopolymers (polyethylene), Copolymers: periodic (agar), random (carrageenan).

Polymer Classification.

- Homopolymers (polyethylene), Copolymers: periodic (agar), random (carrageenan).
- Synthetic (nylon), Natural: proteins (amino-acids), nucleic acids (DNA, RNA), polysaccharides (amylose, cellulose), lignin (plant cement), rubber (fluid of latex cells).

Polymer Classification.

- Homopolymers (polyethylene), Copolymers: periodic (agar), random (carrageenan).
- Synthetic (nylon), Natural: proteins (amino-acids), nucleic acids (DNA, RNA), polysaccharides (amylose, cellulose), lignin (plant cement), rubber (fluid of latex cells).
- linear, branched.

Polymerization

Size of a polymer

the number of constituent monomers (also called the degree of polymerization) may vary from 10^3 up to 10^{10}.

Polymerization

Size of a polymer

the number of constituent monomers (also called the degree of polymerization) may vary from 10^3 up to 10^{10}.

the polymer can arrange itself in many different *spatial configurations*

- can wind around itself to form knots
- can be extended due to repulsive forces between the monomers as a result of excluded-volume

Polymerization

Size of a polymer

the number of constituent monomers (also called the degree of polymerization) may vary from 10^3 up to 10^{10}.

the polymer can arrange itself in many different *spatial configurations*

- can wind around itself to form knots
- can be extended due to repulsive forces between the monomers as a result of excluded-volume
- or can collapse to a ball due to attractive forces.

Targets

- Number of different spatial configurations

Targets

- Number of different spatial configurations
- Typical end-to-end distance (subdiffusive, diffusive or superdiffusive)

Targets

- Number of different spatial configurations
- Typical end-to-end distance (subdiffusive, diffusive or superdiffusive)
- Space-time scaling limit

Targets

- Number of different spatial configurations
- Typical end-to-end distance (subdiffusive, diffusive or superdiffusive)
- Space-time scaling limit
- Average length and height of excursions away from an interface or a surface.

Targets

- Number of different spatial configurations
- Typical end-to-end distance (subdiffusive, diffusive or superdiffusive)
- Space-time scaling limit
- Average length and height of excursions away from an interface or a surface.
- Free energy of the polymer in this limit,

Targets

- Number of different spatial configurations
- Typical end-to-end distance (subdiffusive, diffusive or superdiffusive)
- Space-time scaling limit
- Average length and height of excursions away from an interface or a surface.
- Free energy of the polymer in this limit,
- presence of phase transitions as a function of underlying model parameters

Model Setting: Directed Polymer in Random Environment

Random Polymer

$\mathbf{P}_x$: Probability measure on $(\Omega, \mathcal{F}) := \left((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{P}(\mathbb{Z}^d)^{\otimes \mathbb{N}} \right)$ of sequences $S := (S_n)_{n \geq 0}$ such that:

$$\begin{aligned} &S_0 = x, \\ &\{S_n - S_{n-1}\}_{n \geq 1} \text{ is an IID sequence, and} \\ &\mathbf{P}_x[S_1 = x + e_j] = \mathbf{P}_x[S_1 = x - e_j] = \frac{1}{2d}, \end{aligned} \tag{1}$$

Model Setting: Directed Polymer in Random Environment

Random Environment

IID random variables $\eta := \{\eta_{n,z} : n \in \mathbb{N}, z \in \mathbb{Z}^d\}$, called *the environment*, defined on a probability space $(\Lambda, \mathcal{F}, \mathbb{P})$, that satisfies:

$$\begin{aligned} &\mathbb{E}[\eta_{0,0}] = 0 \text{ and} \\ &\mathbb{E}[\exp(\beta\eta_{0,0})] < \infty, \text{ for all } \beta \in \mathbb{R}. \end{aligned} \tag{2}$$

Polymer Measure

Given $\beta > 0$, $N \in \mathbb{N}$ and a fixed realization of the environment η, we define the measure $\mathbf{P}_N^{\beta,\eta}$ on the space Ω, called the *polymer measure*, by its Radon-Nikodym derivative with respect to $\mathbf{P}_0$:

$$\frac{\mathrm{d}\mathbf{P}_N^{\beta,\eta}}{\mathrm{d}\mathbf{P}_0}(S) = \frac{1}{Z_N^{\beta,\eta}} \exp\left(\beta \sum_{n=1}^{N} \eta_{n,S_n}\right), \tag{3}$$

where $Z_N^{\beta,\eta}$ is the positive normalization factor that makes $\mathbf{P}_N^{\beta,\eta}$ a probability measure.

Known results: Directed Polymer in Random Environment

Bolthausen '87

$$W_N^{\beta,\eta} := \frac{Z_N^{\beta,\eta}}{\mathbb{E}\left[Z_N^{\beta,\eta}\right]}, \tag{4}$$

$$W_\infty^{\beta,\eta} := \lim_{N\to\infty} W_N^{\beta,\eta}, \tag{5}$$

exists $\mathbb{P}$-*a.s.* and is a non-negative random variable.

Known results: Directed Polymer in Random Environment

Bolthausen '87

$$W_N^{\beta,\eta} := \frac{Z_N^{\beta,\eta}}{\mathbb{E}\left[Z_N^{\beta,\eta}\right]}, \tag{4}$$

$$W_\infty^{\beta,\eta} := \lim_{N\to\infty} W_N^{\beta,\eta}, \tag{5}$$

exists $\mathbb{P}$-*a.s.* and is a non-negative random variable.

we have

- *weak disorder* if $W_\infty^\beta > 0$ $\mathbb{P}$-*a.s.* and
- *strong disorder* if $W_\infty^\beta = 0$ $\mathbb{P}$-*a.s.*.

Weak disorder

Polymer paths have the same behavior as the simple random walk (delocalized phase)

Weak disorder

Polymer paths have the same behavior as the simple random walk (delocalized phase)

Comets, Yoshida 2005

Assuming $d \geq 3$ and weak disorder, the measures $\mathbf{P}_N^{\beta,\eta}$, after rescaling, converge in law to the Brownian Motion, for almost all realizations of the environment.

Strong disorder

The polymer is largely influenced by the disorder and is attracted to sites with favorable environment (localize phase).

Strong disorder

The polymer is largely influenced by the disorder and is attracted to sites with favorable environment (localize phase).

Comtets, Shiga, Yoshida 2003

$$\left\{ W_\infty^{\beta,\eta} = 0 \right\} = \left\{ \sum_{n\geq 1} \left(\mathbf{P}_{n-1}^{\beta,\eta} \right)^{\otimes 2} [S_n = S'_n] = \infty \right\} \ \mathbb{P}\text{-a.s.,} \tag{6}$$

where S and S' are two independent polymers with distribution $\mathbf{P}_{n-1}^{\beta,\eta}$. Moreover, if $\mathbb{P}[W_\infty^{\beta,\eta} = 0] = 1$, then there exists some constants $c_1,\ c_2 \in (0,\infty)$ such that,

$$-c_1 \log W_N^{\beta,\eta} \leq \sum_{n\geq 1}^{N} \left(\mathbf{P}_{n-1}^{\beta,\eta} \right)^{\otimes 2} [S_n = S'_n] \leq -c_2 \log W_N^{\beta,\eta} \tag{7}$$

Phase Transition

Comets, Yoshida 2005

There exists a critical value $\beta_c = \beta_c(d) \in [0, \infty]$ with

$$\beta_c = 0 \text{ for } d = 1, 2 \text{ and} \quad (8)$$

$$\beta_c > 0 \text{ for } d \geq 3, \quad (9)$$

such that there is weak disorder for $\beta \in [0, \beta_c)$ and strong disorder for $\beta > \beta_c$.

$\mathcal{L}^2$ region

Example: Showing weak disorder for $d \geq 3$ and small β.

$$\mathbb{E}\left[\left(W_N^{\beta,\eta}\right)^2\right] = \mathbb{E}\mathbf{E}^{\otimes 2}\left[\exp\left(\sum_{i\geq N}\beta\eta_{i,S_i} - \frac{\beta^2}{2} + \beta\eta_{i,S_i'} - \frac{\beta^2}{2}\right)\right] \tag{10}$$

$$\leq \mathbf{E}^{\otimes 2}\left[\exp\left(\sum_i \beta^2 \mathbf{1}_{\{S_i=S_i'\}}\right)\right]. \tag{11}$$

$\mathcal{L}^2$ region

Example: Showing weak disorder for $d \geq 3$ and small β.

$$\mathbb{E}\left[\left(W_N^{\beta,\eta}\right)^2\right] = \mathbb{E}\mathbf{E}^{\otimes 2}\left[\exp\left(\sum_{i\geq N}\beta\eta_{i,S_i} - \frac{\beta^2}{2} + \beta\eta_{i,S_i'} - \frac{\beta^2}{2}\right)\right] \tag{10}$$

$$\leq \mathbf{E}^{\otimes 2}\left[\exp\left(\sum_i \beta^2 \mathbf{1}_{\{S_i=S_i'\}}\right)\right]. \tag{11}$$

Then, $\left\{W_N^{\beta,\eta}\right\}_N$ is uniformly integrable.
$\mathbb{E}W_\infty^{\beta,\eta} = \lim_{N\to\infty}\mathbb{E}W_N^{\beta,\eta} = 1$

Changing the setup

Changing the setup

Our polymer measure

$$\frac{\mathrm{d}\mathbf{P}_N^{\beta,\omega}}{\mathrm{d}\mathbf{P}_0}(S) = \frac{1}{Z_N^{\beta,\omega}}\left(\prod_{n=1}^{N}(1+\beta\omega_{n,S_n})\right), \tag{12}$$

Changing the setup

Our polymer measure

$$\frac{\mathrm{d}\mathbf{P}_N^{\beta,\omega}}{\mathrm{d}\mathbf{P}_0}(S) = \frac{1}{Z_N^{\beta,\omega}} \left(\prod_{n=1}^{N} (1 + \beta\omega_{n,S_n}) \right), \qquad (12)$$

Assuming,

$$\begin{aligned} &\omega_{0,0} \geq -1 \quad \mathbb{P}\text{-a.s.}, \\ &\mathbb{E}[\omega_{0,0}] = 0 \text{ and} \\ &\mathbb{P}[\omega_{0,0} > x] \overset{x\to\infty}{\sim} C_{\mathbb{P}} x^{-\gamma}, \text{ for } \gamma \in (1,2), \end{aligned} \qquad (13)$$

Our results

Free energy, $p(\beta) := \lim_{N\to\infty} \frac{1}{N} Z_N^{\beta,\omega}$

Our results

Free energy, $p(\beta) := \lim_{N\to\infty} \frac{1}{N} Z_N^{\beta,\omega}$

There is a critical value $\gamma_c = \gamma_c(d) := 1 + \dfrac{2}{d}$.

Our results

Free energy, $p(\beta) := \lim_{N\to\infty} \frac{1}{N} Z_N^{\beta,\omega}$

There is a critical value $\gamma_c = \gamma_c(d) := 1 + \dfrac{2}{d}$.

Theorem 1

When the environment satisfies the condition above and if $\gamma \leq \gamma_c$, very strong disorder holds, for all values of $\beta > 0$, in all dimensions $d \geq 1$. In particular, if $d \geq 3$ and $\gamma < \gamma_c$,

$$\lim_{\beta\to 0} \frac{\log|p(\beta)|}{\log\beta} = \alpha, \tag{14}$$

where $\alpha = \alpha(d,\gamma) := \frac{\gamma(\gamma_c - 1)}{\gamma_c - \gamma}$. Also, if $\gamma = \gamma_c$, we have that,

$$\lim_{\beta\to 0} \frac{\log|p(\beta)|}{\log\beta} = \infty. \tag{15}$$

Our results

Our results

Theorem 2

Assuming the same conditions for the environment and if $\gamma > \gamma_c$, we have that $\beta_c > 0$ for dimensions $d \geq 3$.