Scaling limit of a critical random directed graph

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Joint work with Christina Goldschmidt.

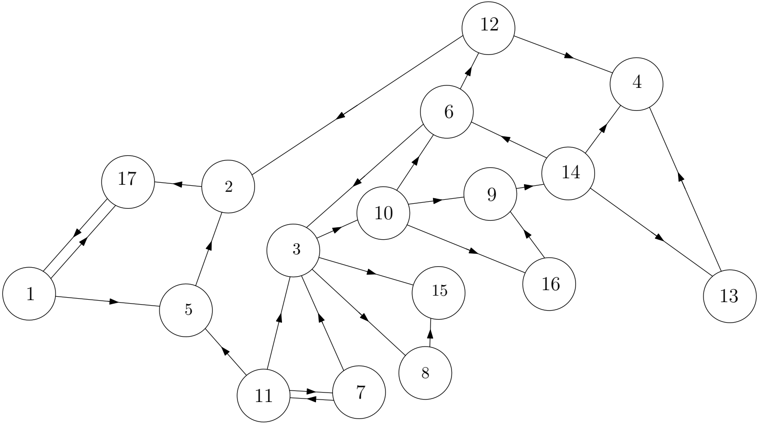
Introduction and main result

Random directed graph

For $n \in \mathbb{N}$ and $p \in [0,1]$, let $\vec{G}(n,p)$ be the random directed defined by :

- Vertices $= \{1, \ldots, n\}$
- Take each of the $n(n-1)$ possible directed edges independently with probability p.

Random directed graph



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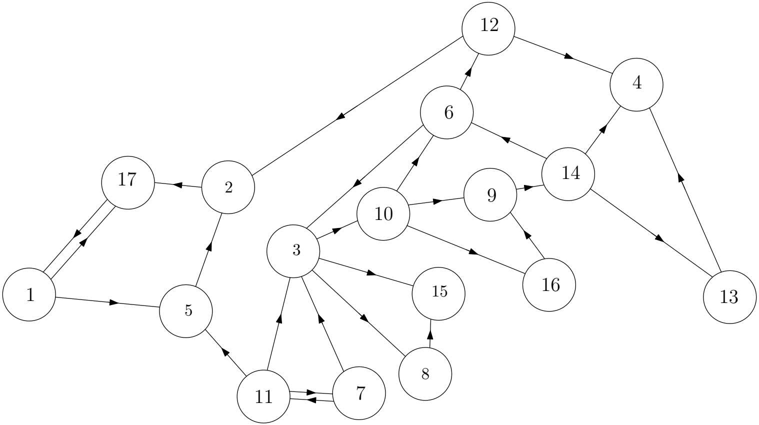
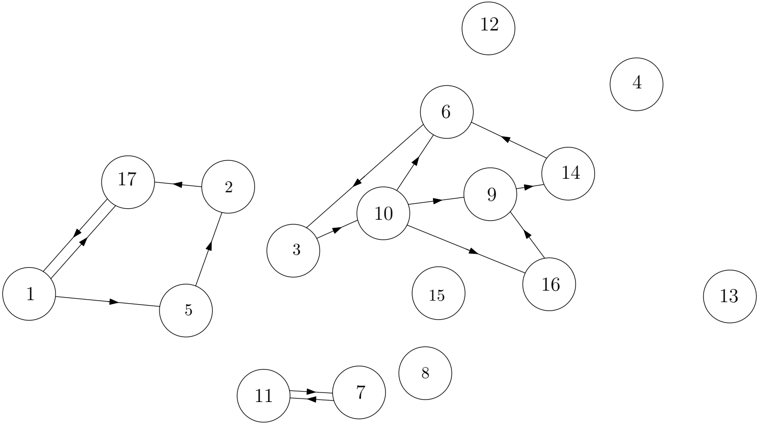
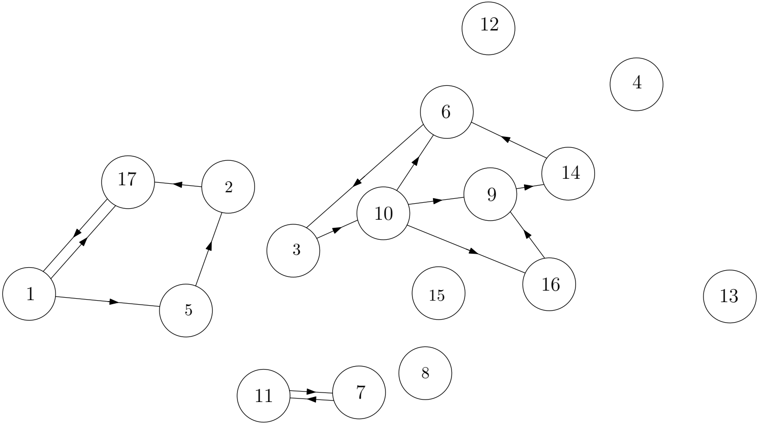


We are interested in the *strongly connected components* : maximal subgraphs where we can go from any vertex to any other in both directions.

Strongly connected components



Strongly connected components



Notice that not all edges are part of a single strongly connected component. Very different from undirected graphs !

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The transition between these two phases can be seen in the so-called *critical window* where $p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$, with $\lambda \in \mathbb{R}$.
We investigate the structure of the components within this window.

Our result : main idea

Let $C_1(n), C_2(n), \ldots$ be the strongly connected components of $\vec{G}(n,p)$, ordered by decreasing sizes. We show that :

- With high probability, the $(C_i(n))$ have no vertices of degree at least 4.
- The number of vertices of degree 3 is of order 1.
- Vertices of degree 3 are linked by vertices of degree 2, the number of which is of order $n^{1/3}$.

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A good idea : view the $(C_i(n))$ as *metric directed multigraphs* (MDM) by removing all vertices of degree 2.

Convergence theorem

Theorem (Goldschmidt-S. '19)

There exists a sequence $\mathcal{C} = (\mathcal{C}_i, i \in \mathbb{N})$ of random strongly connected MDMs such that, for each $i \geq 1$, $\mathcal{C}_i$ is either 3-regular or a loop, and such that

$$\left(\frac{C_i(n)}{n^{1/3}}, i \in \mathbb{N}\right) \xrightarrow{\text{(d)}} (\mathcal{C}_i, i \in \mathbb{N})$$

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This convergence in distribution holds for a strong metric on the set of sequences of MDMs.

Comparison with the Erdős–Rényi graph

Let $G(n,p)$ be the undirected Erdős–Rényi graph, still with $p = 1/n + \lambda n^{-4/3}$. Call $A_1(n), A_2(n), \ldots$ the connected components of $G(n,p)$, ordered by decreasing sizes.

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Theorem

- *(Aldous '97) The sizes of the* $(A_i(n))$ *are of order* $n^{2/3}$.
- *(Addario-Berry, Broutin and Goldschmidt '12) The distances within the* $A_i(n)$ *are of order* $n^{1/3}$. *Specifically, there is a scaling limit of metric spaces :*

$$\left(\frac{A_i(n)}{n^{1/3}}, i \in \mathbb{N}\right) \xrightarrow[\ell^4\text{-}GH]{(d)} (\mathcal{A}_i, i \in \mathbb{N}).$$

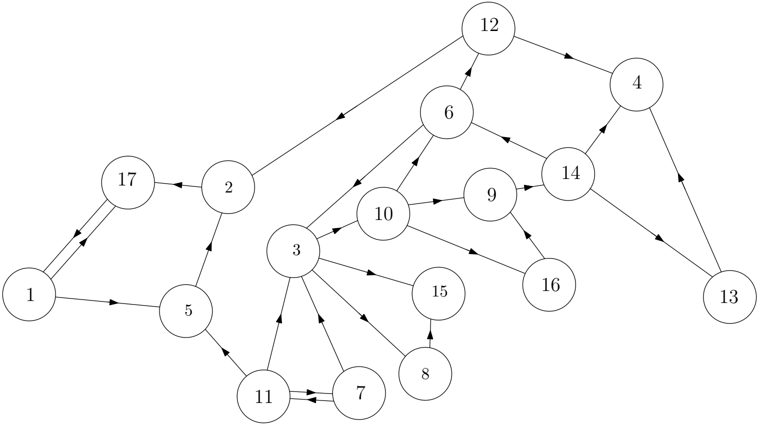
Using an exploration forest

Exploration and a spanning forest

We build a *planar spanning forest* $\mathcal{F}_{\vec{G}(n,p)}$ of $\vec{G}(n,p)$ by using a variant of *depth-first search*.

- Start by classifying 1 as "seen".
- At each step, *explore* the leftmost seen vertex : add to the forest all of its yet unseen outneighbours from left to right with increasing labels, along with their linking edge, and count them as seen.
- If there are no available seen vertices, we take the unseen vertex with smallest label, and put it in a new tree component on the right.

Reminder and practice



Scaling limit of the trees

Let $T_1^n, T_2^n, \ldots$ the trees of $\mathcal{F}_{\vec{G}(n,p)}$, listed by decreasing sizes. We show that

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The limiting trees $(\mathcal{T}_i, i \in \mathbb{N})$ are variants of the celebrated *Brownian continuum random tree.* In particular, they are binary.

Limiting behaviour of the non-tree edges

Edge classification

Remembering that $\mathcal{F}_{\vec{G}(n,p)}$ has a natural *planar ordering*, we can partition the edges of $\vec{G}(n,p)$ into three kinds :

- Edges of $\mathcal{F}_{\vec{G}(n,p)}$.
- "Surplus" edges. These are edges which are not in the forest which point “forwards".
- "Back" edges. These go backwards for the planar structure on the forest.

The interaction between back and forward edges is what creates strongly connected components.

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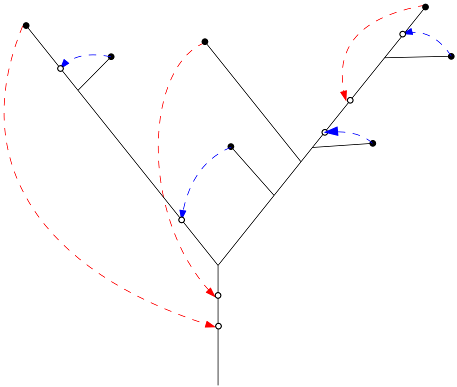
- With high probability, the surplus edges do not contribute to the strongly connected components.
- While the number of back edges does tend to infinity, only a finite number of them contribute to the surplus edges. In fact their start and end points converge in distribution to points of the $\mathcal{T}_i$.

What we end up with



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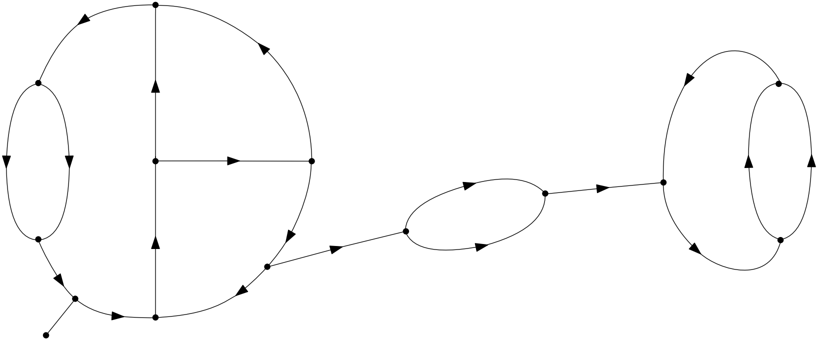

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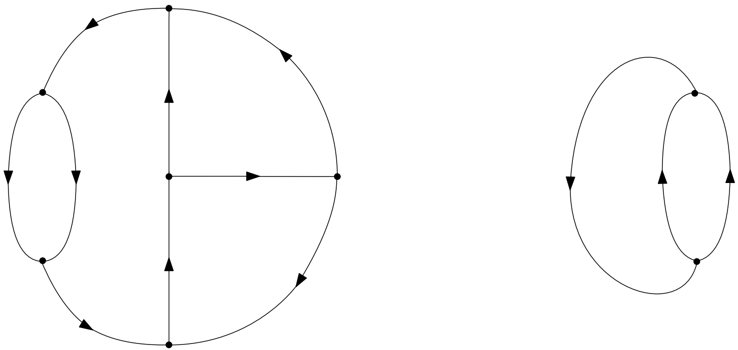

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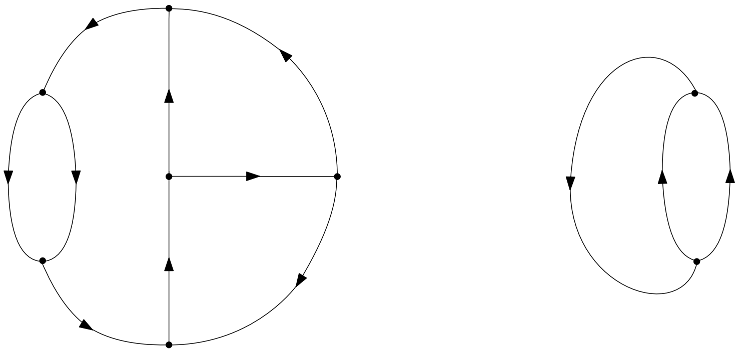

Do this for each tree, and we get the $\mathcal{C}_i$.

Thank you !