Critical values in an anisotropic percolation model

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(Based on joint work with T. Mountford and M.E. Vares)

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Overview

Basic model on $\mathbb{Z}^{1+1}$

Supercritical/ higher dimensions $\mathbb{Z}^{d+1}$ (on going)

Related works

A model investigated by Fontes, Marchetti, Merola, Presutti, Vares:
± 1 spins $\sigma(x, i), x \in \mathbb{Z}, i \in \mathbb{Z}$ s.t.

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  On i-th horizontal level, it follows a Kac potential

$$-\frac{1}{2} J_\gamma(x,y)\sigma(x,i)\sigma(y,i), \sum_{y \neq x} J_\gamma(x,y) = 1,$$

  where $J_\gamma = c_\gamma \gamma J(\gamma(x-y))$ and $J(r), r \in \mathbb{R}$ is smooth and symmetric with support in $[-1,1], J(0) > 0, \int J(r)dr = 1$, c_γ is the normalization constant that tends to 1 as $\gamma \to 0$.

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- Vertical interaction between nearest neighbours with strength $\epsilon(\gamma)$.

  $$-\epsilon\sigma(x,i)\sigma(x,i+1).$$

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Ising $\Rightarrow$ FK percolation with $q = 2$:

$$p(\langle v_1, v_2\rangle) = 1 - e^{-J_\gamma(x,y)} 1_{\langle v_1, v_2\rangle \in E_h} - e^{-2\epsilon(\gamma)} 1_{\langle v_1, v_2\rangle \in E_v}.$$

FK measure:

$$\mathbb{P}(\omega) = \frac{1}{Z(G, p, q)} p^{|\omega|}(1-p)^{|E\setminus\omega|} q^{k(\omega)}.$$

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Graph $\mathbb{Z}^2 = (V, E)$, $V = \{(x, i) : x \in \mathbb{Z}, i \in \mathbb{Z}\}, E = E_h \cup E_v$, s.t.

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- Initial open sites at layer 0: $2N^{2\alpha}$ on $\{-N^{1+\alpha}, \cdots, 0, \cdots, N^{1+\alpha}\}$ with distance $N^{1-\alpha}$

Picutre

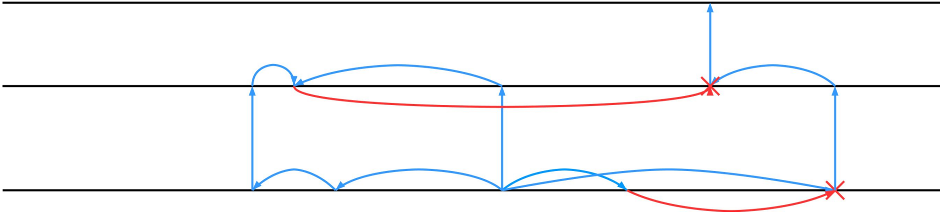

Figure: Anisotropic percolation on $\mathbb{Z}^2$

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  where $N_k(x) = \sum_{y \sim x} \hat{\xi}_k(y), \eta^w_{k+1} \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(1/2N)$.

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- $\hat{\xi}_{k+1}(x) =$

  $$\hat{\xi}_k(x) + \text{ Laplacian } + \text{ martingale } - \underbrace{\frac{1}{2N}\sum_{y\sim x}\hat{\xi}_k(y)\sum_{j\le k}\hat{\xi}_j(x)}_{\text{converges when } \alpha=1/5} + \text{ Error }.$$

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- Total attrition is $N^{5\alpha-1}$

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Theorem 1 (Mountford, Vares, X.).

The critical value is $b = 2/5$ ($\epsilon(N) = \kappa N^{-b}$): there exist positive constants C_1 and C_2 such that for $\kappa < C_1$, there is no percolation and for $\kappa > C_2$, the percolation appears.

Supercritical/ higher dimensions cases (on going)

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- When $\lambda > 1, d = 1$ (horizontal open probability is $\lambda/2N$): the critical vertical interaction is $\epsilon(N) = e^{-\kappa N}$.
- When $\lambda > 1, d > 1$, there always exists a percolation.
  - What is the probability of a large but finite size cluster?

Thanks!