LLDP for compound Poisson process with catastrophes

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Scaling.

We are interested in the Local Large Deviation Principle (LLDP) for the family of processes

$$\xi_T(t) := \frac{\xi(Tt)}{T},\ t \in [0,1],$$

where $T \to \infty$ is an increasing parameter.

Trajectories of the process $\xi_T(\cdot)$ almost surely (a.s.) belong to the space $\mathbb{D}[0,1]$ of cádlág functions. Let

$$\rho(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|$$

for $f, g \in \mathbb{D}[0,1]$.

Definition of Local Large Deviation Principle.

A family of random processes $\xi_T(\cdot)$ satisfies LLDP on the set $G \subset \mathbb{D}[0,1]$ with a rate function $I = I(f) : \mathbb{D}[0,1] \to [0,\infty]$ and the normalizing function $\psi(T)$ such that $\lim_{T\to\infty} \psi(T) = \infty$, if for any function $f \in G$ the following equalities hold

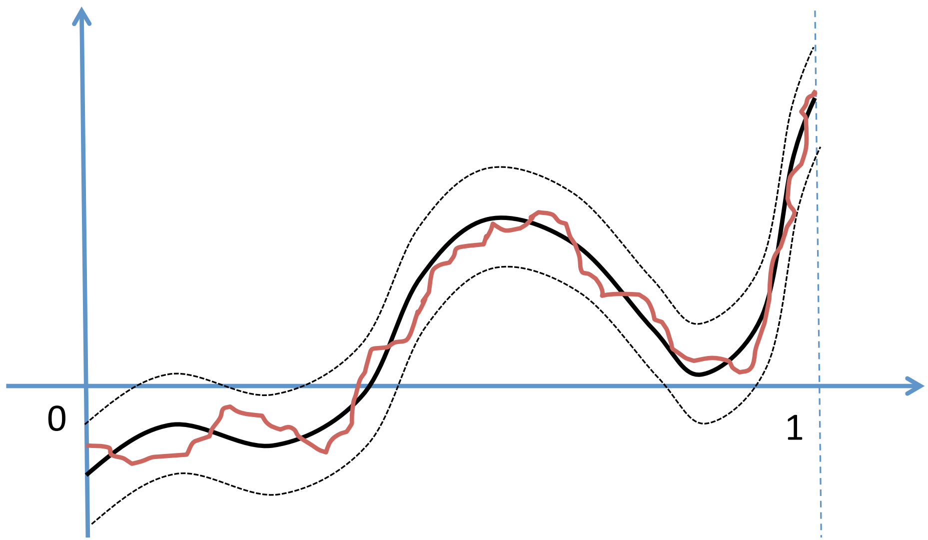
$$\lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\xi_T(\cdot) \in U_\varepsilon(f))$$
$$= \lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\xi_T(\cdot) \in U_\varepsilon(f)) = -I(f),$$

where $U_\varepsilon(f)$ stands for ε-neighborhood of f,

$$U_\varepsilon(f) := \{g \in \mathbb{D}[0,1] : \ \rho(f,g) < \varepsilon\}.$$

Local Large Deviation Principle.

$$\mathbf{P}\Big((\xi_T(t))_{t\in[0,1]} \approx (f(t))_{t\in[0,1]}\Big) \approx \exp\Big(-\psi(T)I(f)\Big)$$



Definition of Large Deviation Principle.

A family of random processes $\xi_T(\cdot)$ satisfies LDP on metric space $(\mathbb{D}[0,1],\rho)$ with a rate function $I = I(f) : \mathbb{D}[0,1] \to [0,\infty]$ and the normalizing function $\psi(T)$ such that $\lim\limits_{T\to\infty} \psi(T) = \infty$, if, for any $c \geq 0$ the set $\{f \in \mathbb{D}[0,1] : I(f) \leq c\}$ is a compact set on $(\mathbb{D}[0,1],\rho)$ and, for any set $B \in \mathfrak{B}_{(\mathbb{D}[0,1],\rho)}$ the following inequalities hold:

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\xi_T(\cdot) \in B) \leq -I([B]) := -\inf_{f\in[B]} I(f),$$

$$\liminf_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\xi_T(\cdot) \in B) \geq -I((B)) := -\inf_{f\in(B)} I(f),$$

where $\mathfrak{B}_{(\mathbb{D}[0,1],\rho)}$ is the Borel σ-algebra constructed by open cylindrical subsets of the space $\mathbb{D}[0,1]$, $B \in \mathfrak{B}_{(\mathbb{D}[0,1],\rho)}$, $I(\emptyset) = \infty$.

Random processes with catastrophes. Motivations:

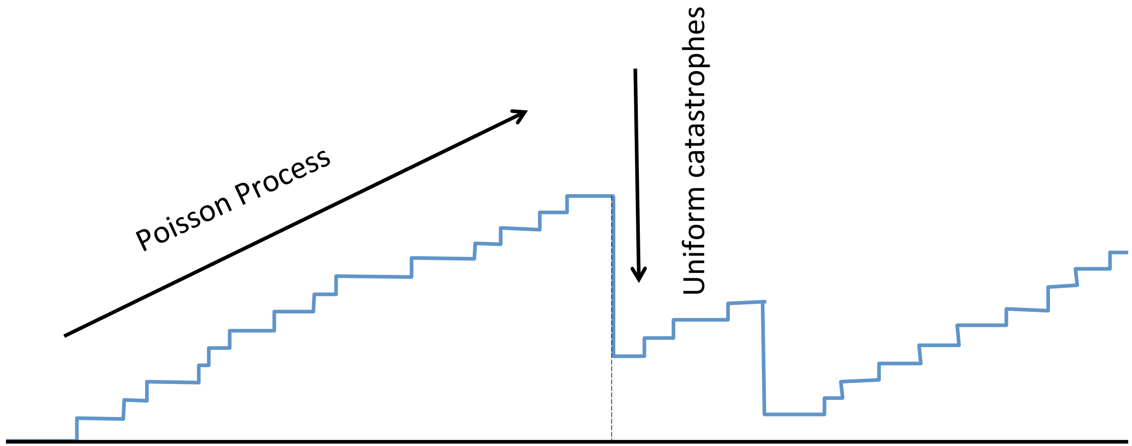
1. population dynamic;
2. queueing systems;
3. price spread dynamics;
4. search strategies;
5. ...

Random processes with catastrophes.

“original” processes	catastrophes	catastrophes occurrence
Poisson processes	total	Markov processes =
birth-death processes	uniform	= exponential rate
Wiener processes	binomial	renewal processes
. . .	. . .	. . .

Our model.

"original" processes	catastrophes	catastrophes occurrence
Poisson processes	total	Markov processes =
birth-death processes	(almost) uniform	= exponential rate
Wiener processes	binomial	renewal processes
...	...	...



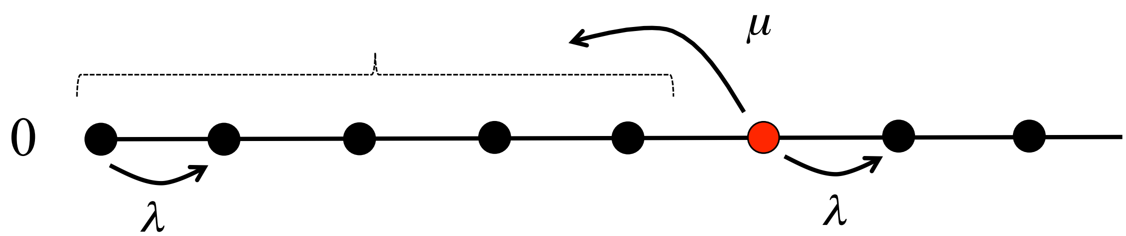
Our model.

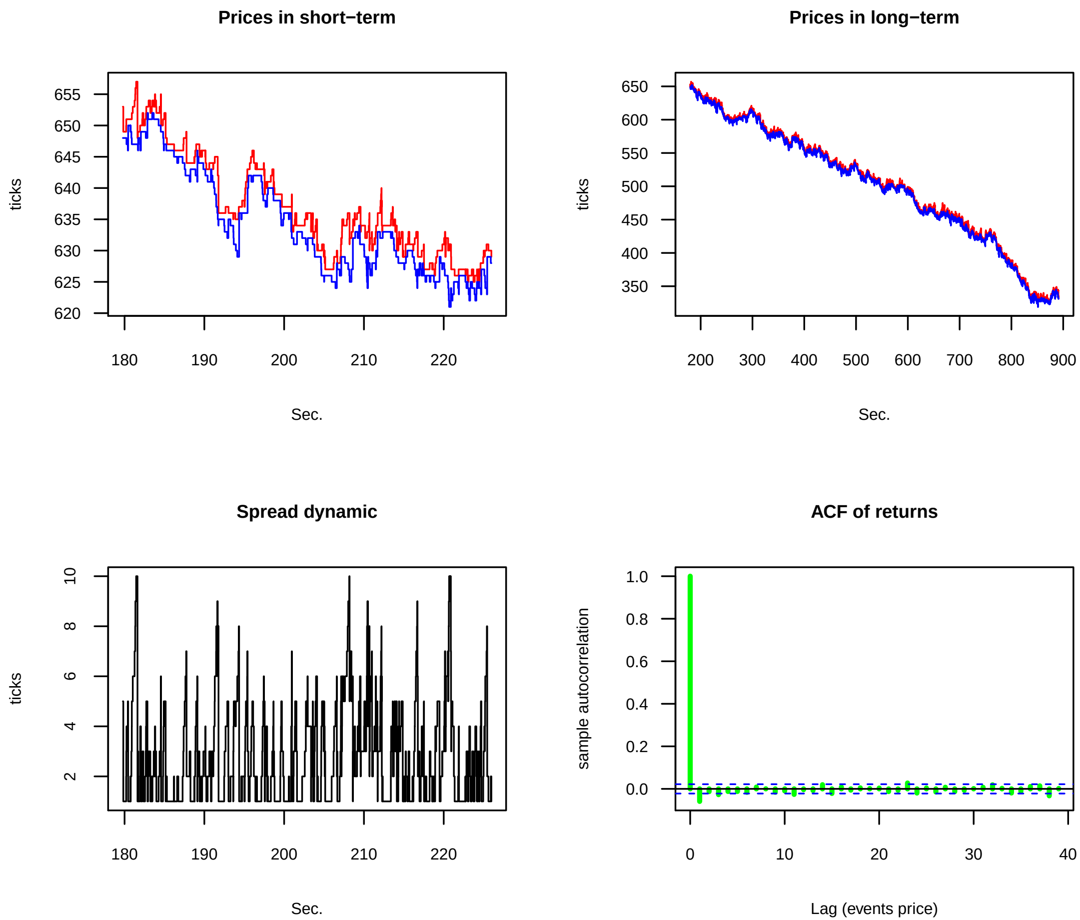
Let us start by definition of the process. We construct the process in two steps. First, we define the discrete time Markov chain $\eta(k)$ with state space $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ and transition probabilities

$$\mathbf{P}(\eta(k+1) = j|\eta(k) = i) = \begin{cases} \frac{\lambda}{\lambda+\mu}, & \text{if } j = i+1, \\ \frac{\mu}{i(\lambda+\mu)}, & \text{if } 0 \le j < i, i \ne 0, \\ 1, & \text{if } j = 1, i = 0, \end{cases}$$

where λ and μ are positive constants. Let $\eta(0) = 0$. Second, let $\nu(t)$, $t \in \mathbb{R}^+$ be Poisson point process with rate α, which does not depend on the chain $\eta(\cdot)$. Define

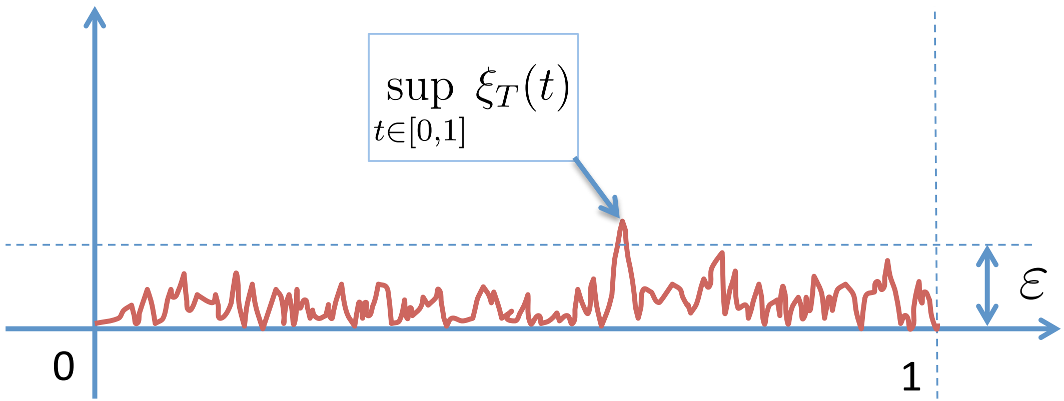
$$\xi(t) := \eta(\nu(t)), \ t \in \mathbb{R}^+.$$




Prices in short−term
ticks
655
650
645
640
635
630
625
620
180 190 200 210 220
Sec.
Prices in long−term
ticks
650
600
550
500
450
400
350
200 300 400 500 600 700 800 900
Sec.
Spread dynamic
ticks
2 4 6 8 10
180 190 200 210 220
Sec.
ACF of returns
sample autocorrelation
1.0
0.8
0.6
0.4
0.2
0.0
0 10 20 30 40
Lag (events price)

Theorem 1. For every $\varepsilon > 0$

$$\mathbf{P}\left(\lim_{T\to\infty} \sup_{t\in[0,1]} \xi_T(t) > \varepsilon\right) = 0.$$



Variation of functions. Let

$\mathbb{AC}_0^M[0,1]$ is the set of monotonically nondecreasing, absolutely continuous on the interval $[0,1]$ functions that start from zero;

$\mathbb{AC}_0^+[0,1]$ is the set of absolutely continuous on the interval $[0,1]$ functions starting from zero and taking positive values for $t \in (0,1]$;

$\mathrm{Var} f_{[0,a]}$ is a total variation of the function f on the interval $[0,a]$

Every function $f \in \mathbb{AC}_0^+[0,1]$ can be uniquely represented as a difference of functions $f^+ \in \mathbb{AC}_0^M[0,1]$ and $f^- \in \mathbb{AC}_0^M[0,1]$ such that

$$\mathrm{Var} f_{[0,1]} = \mathrm{Var} f^+_{[0,1]} + \mathrm{Var} f^-_{[0,1]}.$$

Nondecreasing functions f^+ and f^- are called respectively positive and negative variations of the function f.

Theorem 2. Let B_f be the set of monotonically nondecreasing functions g, such that for almost all $t \in [0,1]$ the inequality $\dot g(t) \geq \dot f^+(t)$ holds.

A family of random processes $\xi_T(\cdot)$ satisfies the LLDP on the set $\mathbb{AC}_0^+[0,1]$ with the normalizing function $\psi(T) = T$ and the rate function

$$I(f) = \frac{\alpha\mu}{\lambda+\mu} + \inf_{g \in B_f} \int_0^1 \left(\dot g(t) \ln\left(\frac{\dot g(t)(\lambda+\mu)}{\alpha\lambda} \right) - \dot g(t) + \frac{\alpha\lambda}{\lambda+\mu} \right) dt.$$

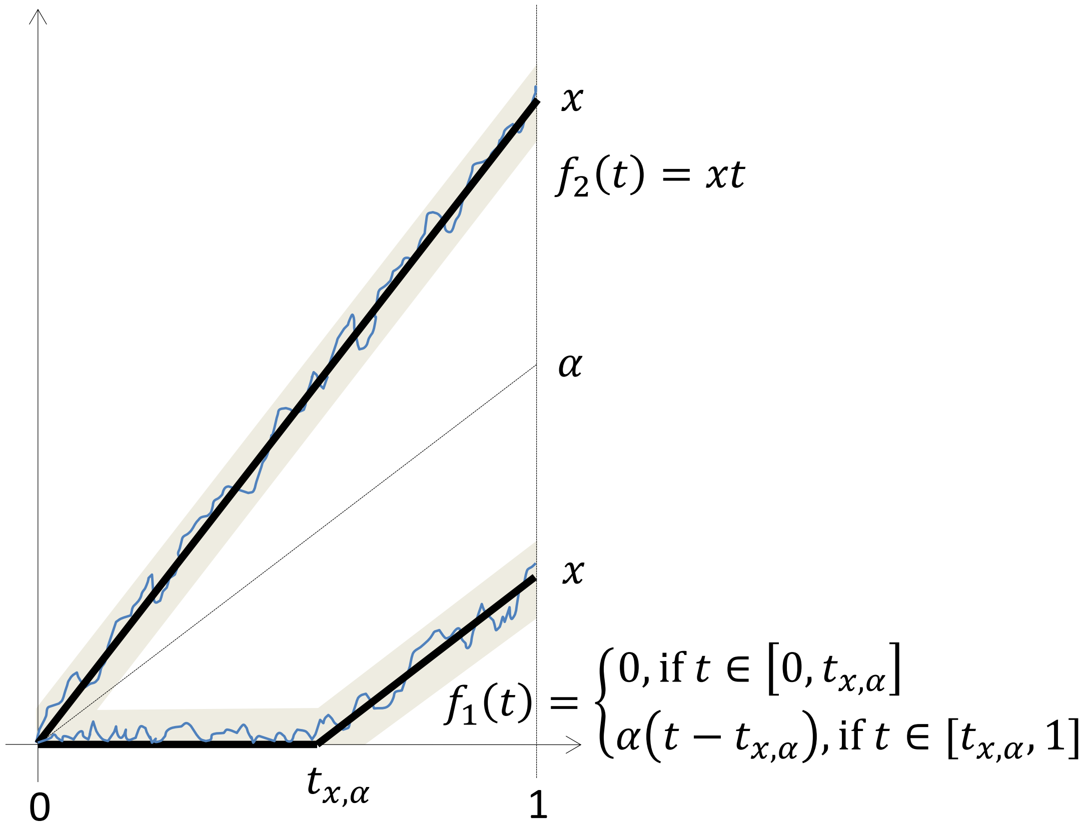
Remark. The rate function can be rewritten:

$$I(f) = \frac{\alpha\mu}{\lambda+\mu} + \int_0^1 \left(\dot f^+(t) \ln\left(\frac{\dot f^+(t)(\lambda+\mu)}{\alpha\lambda} \right) - \dot f^+(t) + \frac{\alpha\lambda}{\lambda+\mu} \right) \times$$
$$\times \mathbf{I}\left(\dot f^+(t) \geq \frac{\alpha\lambda}{\lambda+\mu} \right) dt.$$

Theorem 3. (LDP for $\xi_T(1)$) The family of random variables $\xi_T(1)$ satisfies LDP with normalized function $\psi(T) = T$ and with rate function

$$I(x) = \begin{cases} \infty, & \text{if } x \in (-\infty, 0), \\ x \ln\left(\frac{\lambda+\mu}{\lambda}\right), & \text{if } x \in [0, \alpha), \\ x \ln\left(\frac{x(\lambda+\mu)}{\alpha\lambda}\right) - x + \alpha, & \text{if } x \in [\alpha, \infty). \end{cases}$$

Logachov, A., Logachova, O. and Y.A. Large deviation in population dynamics with catastrophes. *Statistics & Probability Letters*, v.149, pp. 29–37, 2019.


x
$f_2(t) = xt$
α
x
$f_1(t) = \begin{cases} 0, \text{if } t \in [0, t_{x,\alpha}] \\ \alpha(t - t_{x,\alpha}), \text{if } t \in [t_{x,\alpha}, 1] \end{cases}$
0
$t_{x,\alpha}$
1

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References:

Logachev, A., Logacheva, O., Yambartsev, A. (2019) The local principle of large deviations for compound Poisson process with catastrophes. arXiv preprint arXiv:1806.07459v2.

Thank you