The long range divisible sandpile

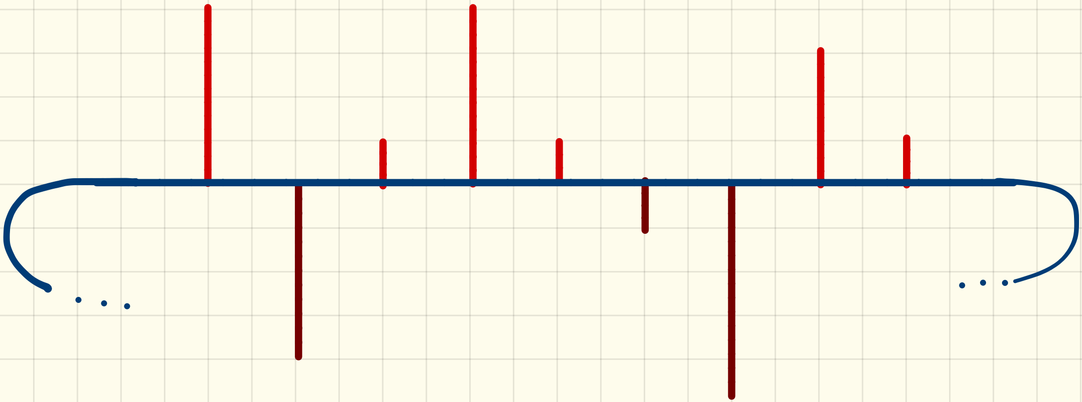
L Chiarini
IMPA - TU DELFT

j.w.w. M. Jara, W. Ruszel
IMPA TU DELFT

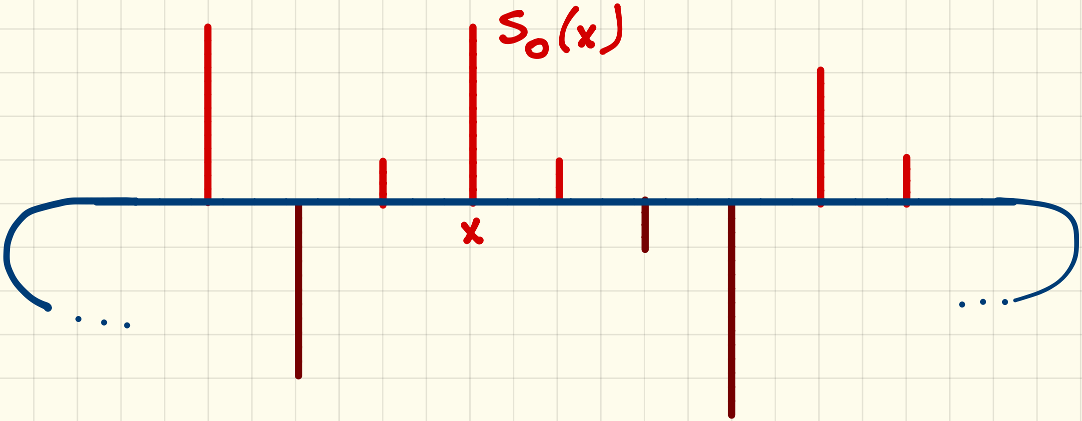
EBP - 2019

$G = \mathbb{Z}_n^d / \sim$ Discrete Torus

$s_0 : \mathbb{Z}_n^d \longrightarrow \mathbb{R}$ Dist. of Mass (or holes)



Deterministic diffusion of mass



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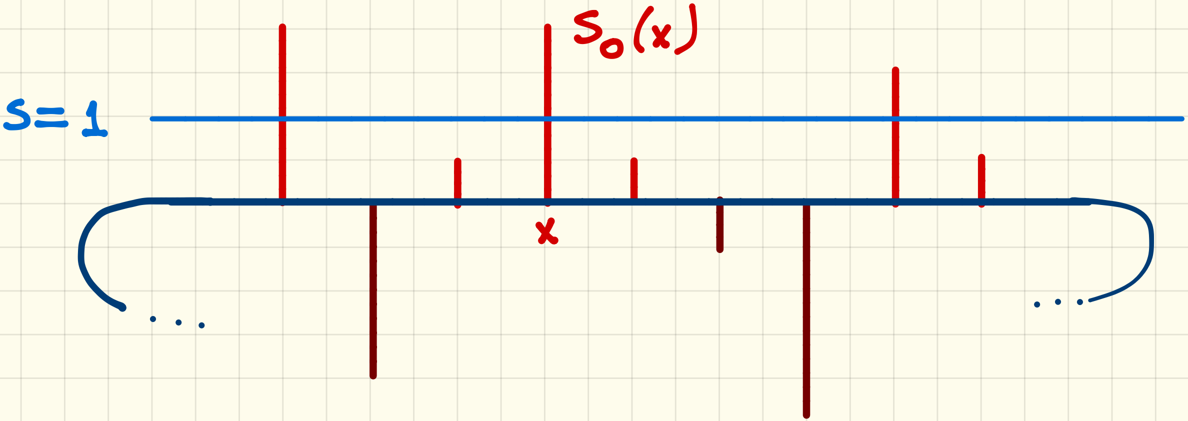
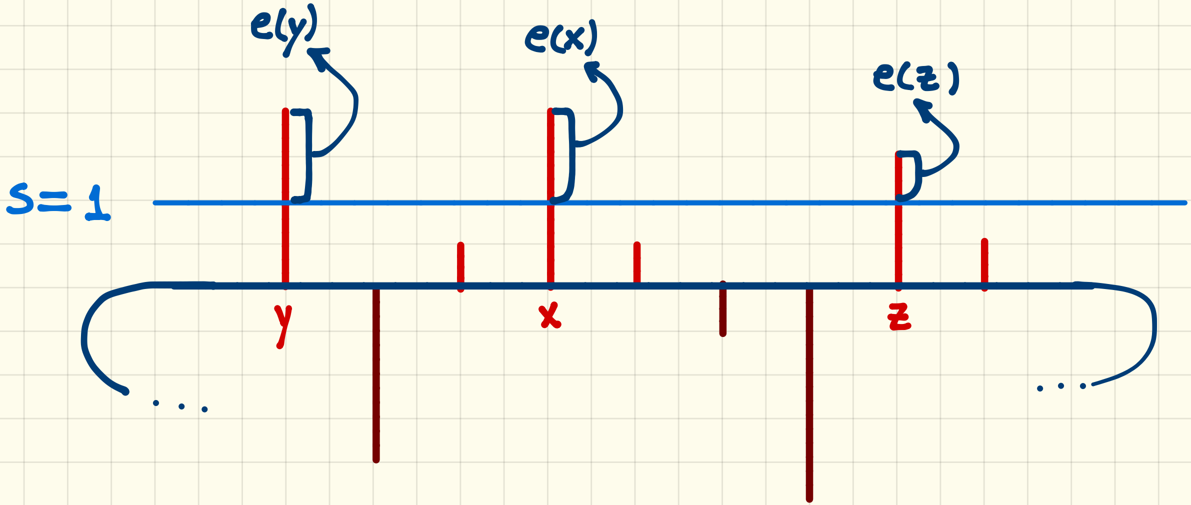


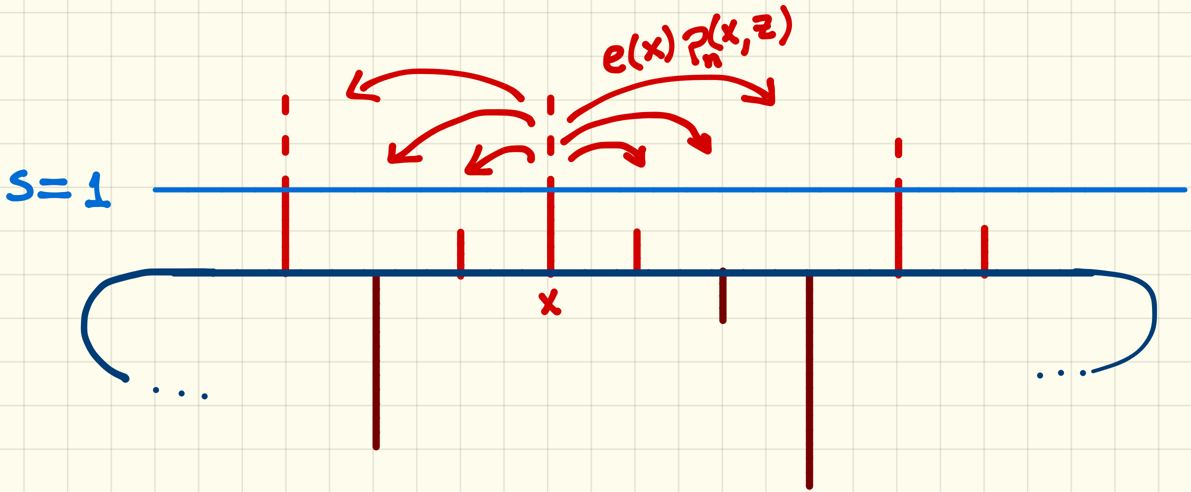
Deterministic diffusion of mass

Spread the excess $e(x) = (s(x) - 1)^+$



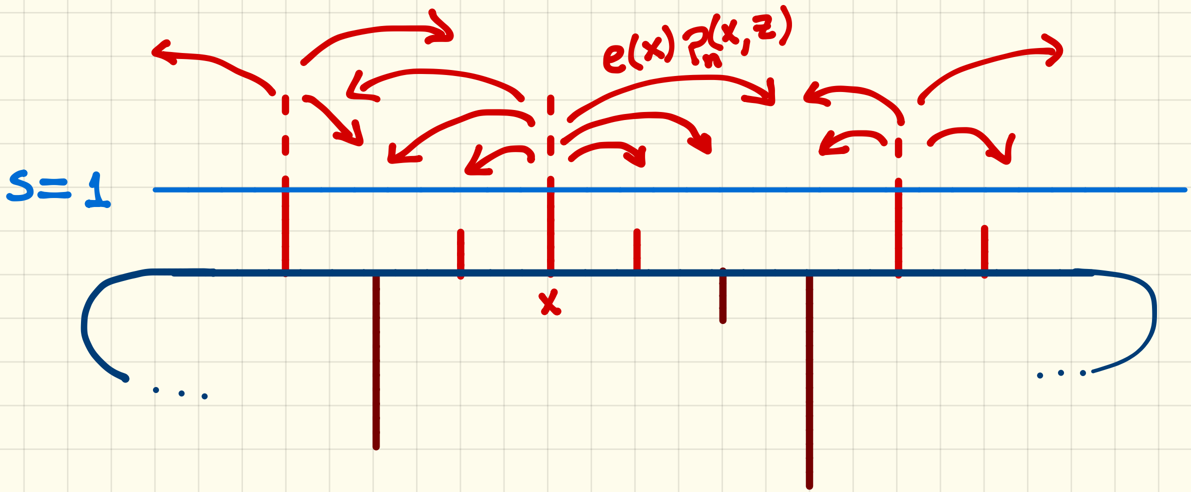
Deterministic diffusion of mass

Spread the excess $e(x) = (s(x)-1)^+$ according to $P_n(x,\cdot)$ of a random walk.



Deterministic diffusion of mass

Spread the excess $e(x) = (s(x)-1)^+$ according to $P_n(x,\cdot)$ of a random walk simutaneously.



We will consider

$$P_n(x,y) = P_n^{\alpha}(x-y,0) \quad \text{and}$$

$$P_n(0,x) = \sum_{\substack{z \in \mathbb{Z}^d \setminus \{0\} \\ z \equiv x \ (\mathrm{mod}\ \mathbb{Z}_n^d)}} \frac{c(d,\alpha)}{\|z\|_2^{d+\alpha}}$$

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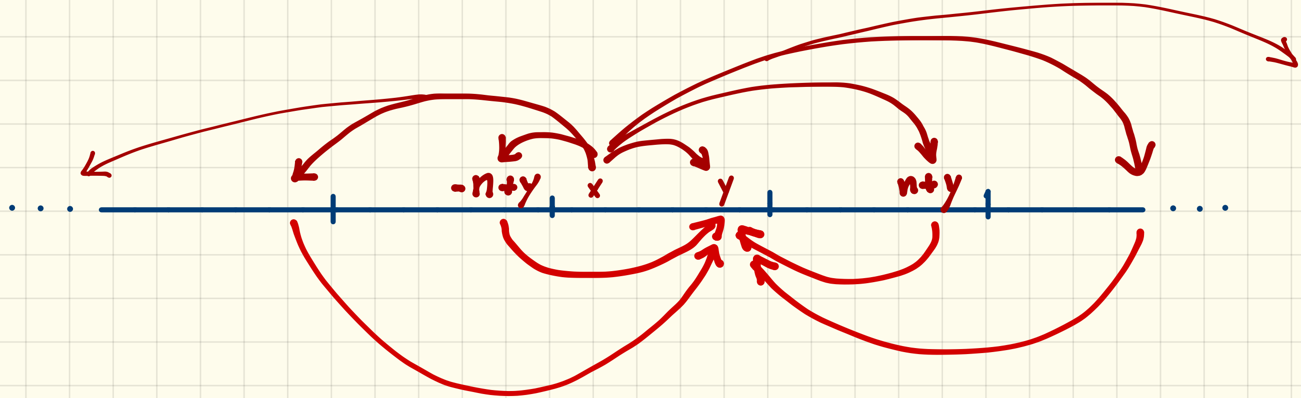
→ Norm. Const.



The Odometer

$$u_t(x) \overset{\text{def}}{=} \sum_{j=0}^{t} (s_j(x) - 1)^+$$

= total mass expelled by x up to time t.

Satifies

$$s_t = s_0 - (-\Delta)_\eta^{\alpha/2} u_t$$

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$\to e_t(x)$

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Satifies

generator of p_n^α

$$s_t = s_0 - (-\Delta)_n^{\alpha/2} u_t$$

Explosion vs Stabilisation

$$u_\infty(x) = \lim_{t\to\infty} u_t(x)$$

$$\begin{cases} u \equiv +\infty & \Rightarrow \text{Explosion} \\ u < +\infty & \Rightarrow \text{Fixation} \end{cases}$$

For $s_0 : \mathbb{T}_n^d \longrightarrow \mathbb{R}$ s.t $\sum_x s_0(x) = n^d$

we have that $u_\infty < +\infty$ and

$$s_\infty \equiv \lim_{t\to\infty} s_t(x) \equiv 1.$$

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? What happens if we take $(s_0(x))_{x\in\mathbb{T}_n^d}$ "almost" iid ?

Let $(\sigma(x))_{x\in\mathbb{T}_n^d}$ iid with Var $\sigma < +\infty$?

And set

$$S_0(x) = 1 + \sigma(x) - \frac{1}{n^d} \sum_{y\in\mathbb{T}_n^d} \sigma(y)$$

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In which case

$$\begin{cases} (-\Delta)_n^{\alpha/2} u_\infty^n = 1 - s_0(x) = -\sigma_0(x) + \frac{\sum \sigma(y)}{n^d} \\ \min_x u_\infty^n(x) = 0 \end{cases}$$

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Then, we want to know

1. Asymptotics of $E[u_\infty^n]$ in the case that $\sigma \sim N(0,1)$.

2. Scaling of the field u^n.

$$\Xi_n(x) = a_\alpha(n) \sum_{z \in \mathbb{T}_n^d} u_\infty^n(nz)\, 1_{B^\infty(z, \frac{1}{2n})}$$

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$$E[u_n^\alpha] \asymp \Phi_{d,\gamma}(n) = \begin{cases} n^{\gamma - d/2}, & \gamma > \frac{d}{2} \\ \log n, & \gamma = d/2 \\ (\log n)^{1/2}, & \gamma < \frac{d}{2} \end{cases}$$

(Levine, Murucan, Peres, Ugurcan - 15) : n.n.

(C., Jara, Ruszel - 18'). long-range

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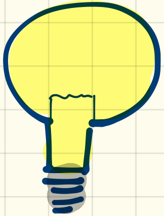

"Proof"

Talagram Chaining Inequality

+

Rate of Convergence of eigenvalues

2. Scaling of the field u^n.

$$\Xi_n(x) = a_\alpha(n) \sum_{z \in \mathbb{T}_n^d} u_\infty^n(nz)\, \mathbb{1}_{B^\infty(z, \frac{1}{2n})}$$

$$a_\alpha(n) = \begin{cases} n^{\frac{d-2\gamma}{2}} &, \alpha \neq 2 \\ n^{\frac{d-4}{2}} \cdot \log n &, \alpha = 2 \end{cases}$$

Then, $\Xi_n \longrightarrow \Xi_{2\gamma}$ the the 2γ-FGF (Fractional G. Field)

(Cipriani, Hazra, Ruszel - 17): n.n

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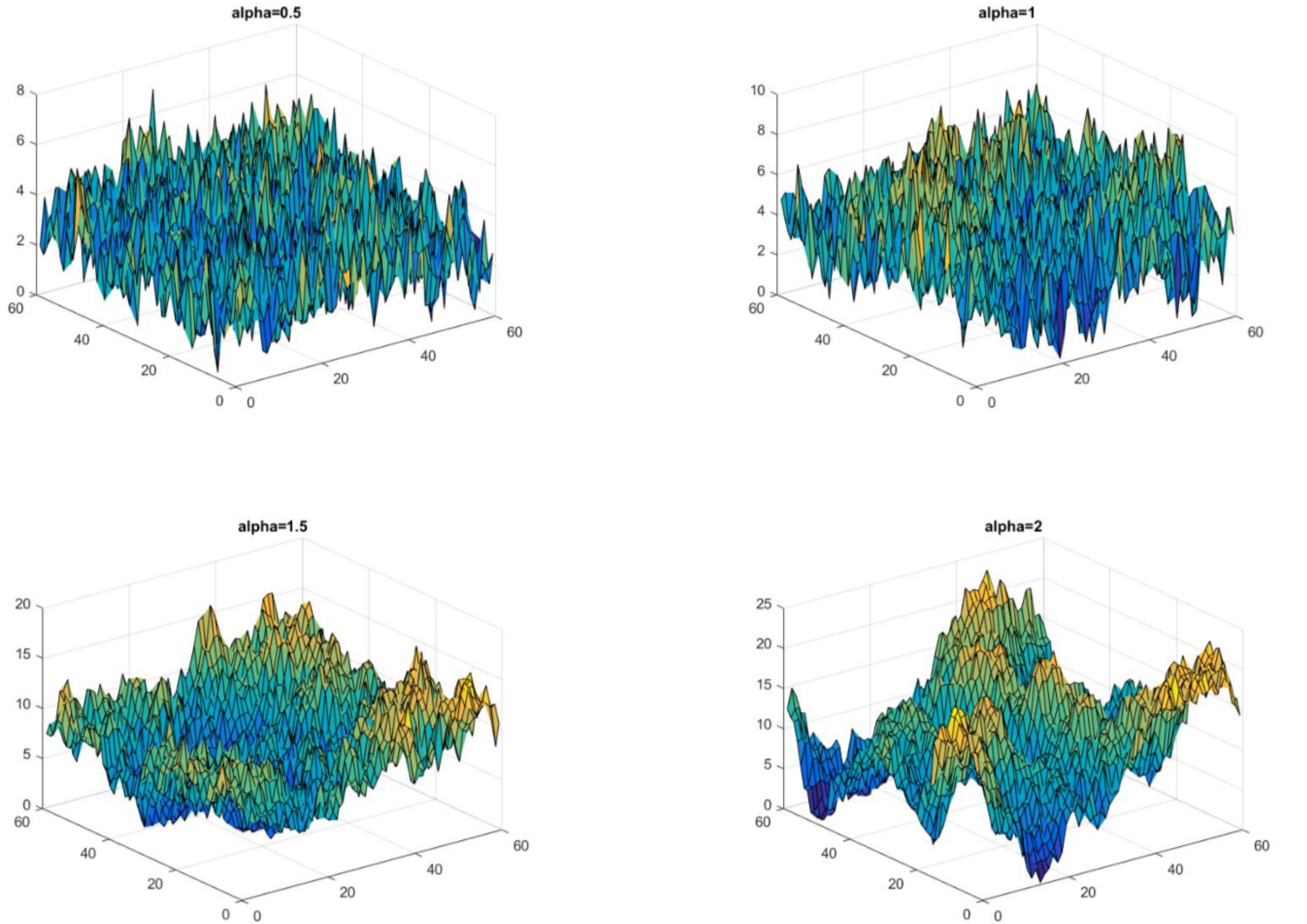
That is for all $f \in C^\infty(\mathbb{T})$ s.t $\int_{\mathbb{T}} f\, dx = 0$

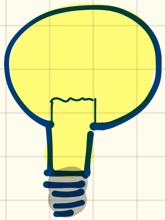
$$\langle \Xi_{2r}, f \rangle \sim N(0, \|f\|_{-2r})$$

with

$$\|f\|_{-2r} = \sum_{k \in \mathbb{Z}^d_*} \frac{|\hat{f}(k)|^2}{k^{2r}}$$

$$= \langle f, (-\Delta)^r f \rangle_{L^2}$$


alpha=0.5
alpha=1
alpha=1.5
alpha=2
0
2
4
6
8
10
5
15
20
25
40
60



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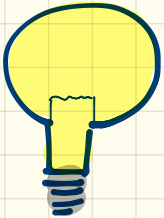
Tightness: Sobolev + Chebchev

Finite dim convergence : Moment methods + Eigenvalues

Thanks



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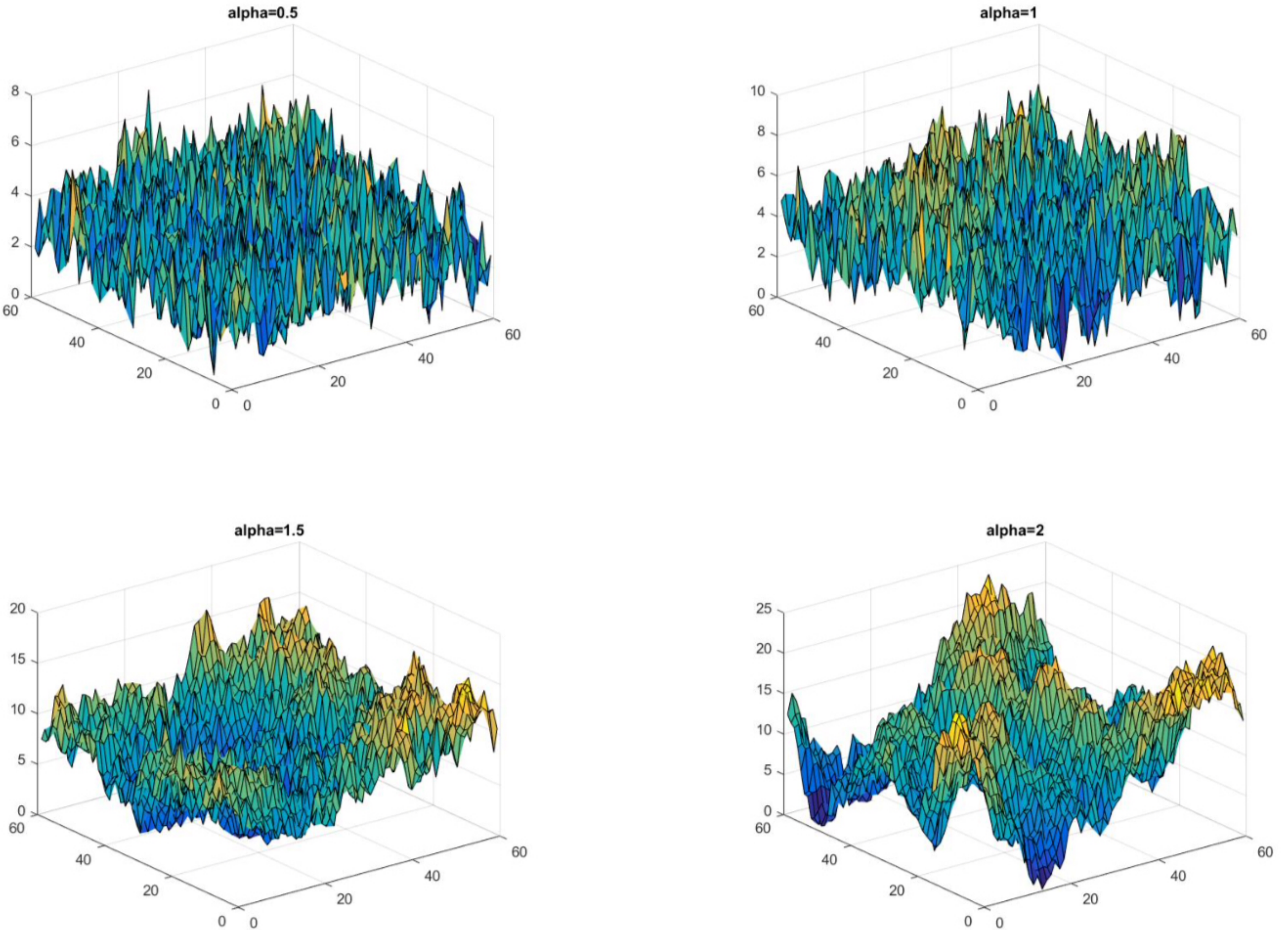
alpha=0.5
0
2
4
6
8
60
40
20
0
0
20
40
60
alpha=1
0
2
4
6
8
10
60
40
20
0
0
20
40
60
alpha=1.5
0
5
10
15
20
60
40
20
0
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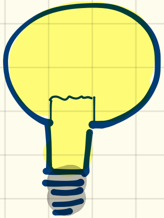

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<u>scaling</u> <u>is</u> <u>the</u> <u>same</u>

Let $(\sigma(x))_{x\in\mathbb{T}_n^d}$ ~~iid~~ with $\mathrm{Var}\,\sigma < +\infty$?

normal and with covariance

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And set

$$S_o(x) = 1 + \sigma(x) - \frac{1}{n^d}\sum_{y\in\mathbb{T}_n^d}\sigma(y)$$

for $p(x,y) \sim$ SRW

Scaling gives fields smother than 4-FGG, but not rougher

(Cipriani, de Graff, Ruszel - 18)

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Let $(\sigma(x))_{x\in\mathbb{T}_n^d}$ iid with ~~$\mathrm{Var}\,\sigma<+\infty$?~~

heavy tailed with decay $\|\cdot\|^{-\beta}$

And set

$$S_0(x) = 1+\sigma(x) - \frac{1}{n^d}\sum_{y\in\mathbb{T}_n^d}\sigma(y)$$

for $p(x,y)\sim$ SRW

Scaling results in non-Gaussian fields

$$E\left[\exp\left(i\langle \square, f\rangle\right)\right] = \exp\left(-\|f\|_{-4}^{\beta}\right)$$

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For $s_0 : \mathbb{T}_n^d \longrightarrow \mathbb{R}$ [redacted]

[redacted]

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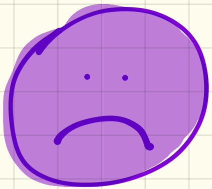
No idea



Thanks

Again