Interacting diffusions on random sparse graphs

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Joint work with
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- My purpose in this talk is to show how we go beyond the mean-field case.
- Remark: everything is loosely stated. I'm also going to present everything in a particular example.
- First I will introduce the Kuramoto Model.

The Kuramoto model

The Kuramoto model is the following system of ODEs ($t \in [0, T]$) :

$$\mathrm{d}\theta_i(t) = \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j(t) - \theta_i(t))\mathrm{d}t + \omega_i \mathrm{d}t, \; i \in \{1, \cdots, N\}$$

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- Mean-field equals complete graph.

Motivation

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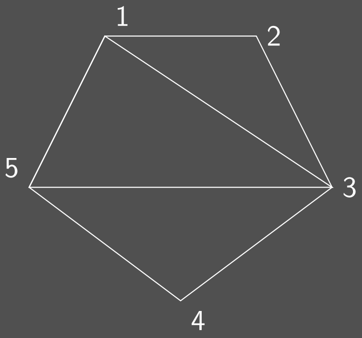
Since our spins live on $\mathbb{R}$ (and not in $\mathbb{S}^1$) we need that $t \in [0, T]$.
Our contribution is to remove the mean-field assumption.

Interacting particles on graphs



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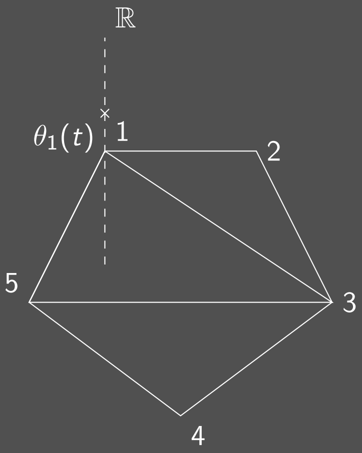

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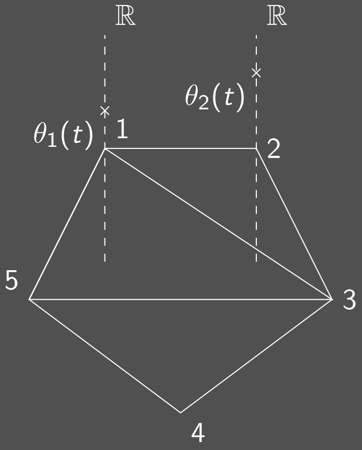

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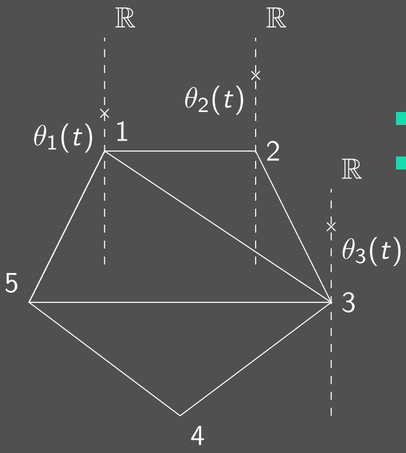

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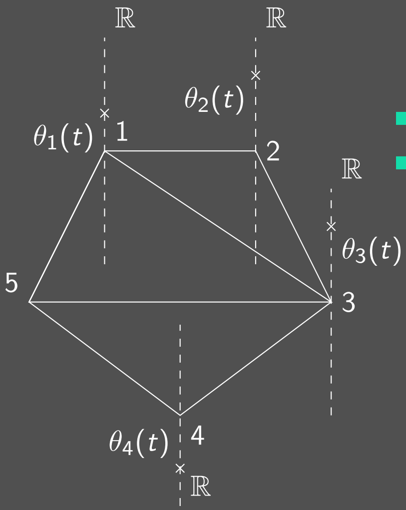

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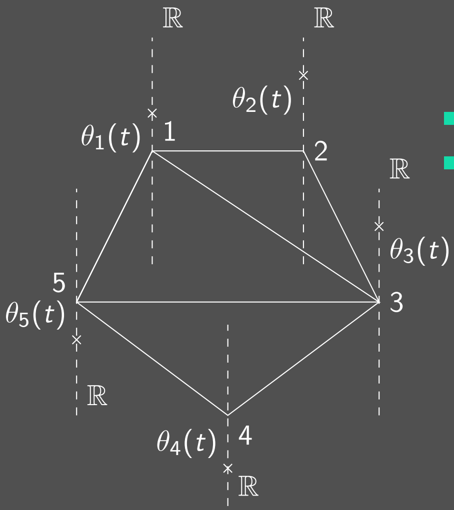

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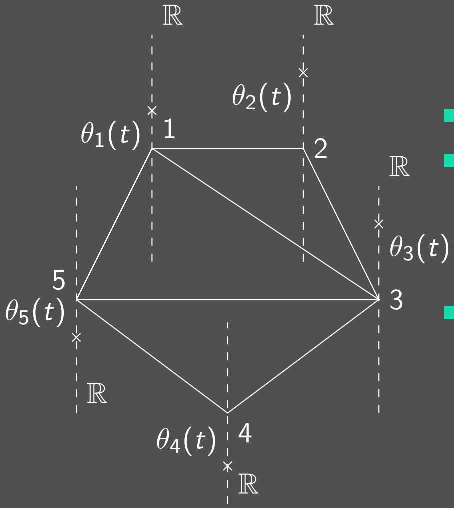

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- Think that each particle is on a vertex. It is like a spin in $\mathbb{R}$ evolving in time.
- We want the Kuramoto model with interactions given by G.

The stochastic Kuramoto model on graphs

The particles follow the following system of SDEs ($t \in [0, T]$):

$$\mathrm{d}\theta_i(t) = \frac{1}{d_i} \sum_{j \sim_G i} \sin(\theta_j(t) - \theta_i(t))\mathrm{d}t + \omega_i \mathrm{d}t + \mathrm{d}B_i(t), \, \forall i \in G.$$

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- What do we want to prove?

Law of Large Numbers

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GOAL: To prove that L_{G_N} converges to something, possibly satisfying LDP. We also want to see how the limit depends on the sequence $(G_N)_{N\in\mathbb{N}}$.

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Summing up: we have SLLN with exponential rates of convergence and a nice description of the limit object.

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$$\lim_{N\to\infty} L_{G_N} \text{ is determined by either } \lim_{N\to\infty} Np(N) = \infty \text{ or } \lim_{N\to\infty} Np(N) = c < \infty.$$

Our contributions

$\lim_{N\to\infty} Np(N)$	$= +\infty$	$= c < +\infty$
G_N "looks like"	K_N	$GW(c)$ tree
In the sense	A_{K_N} and A_{G_n} are "close" very nicely	Benjamini-Schramm
$L_{G_N} \to$??	McKean-Vlasov	New object
Limit Theorem	LDP	Hydrodynamic limit (SLLN) Propagation of Chaos
Sync?	??	NO (through simulations)

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- We are NOT assuming any growth condition on the divergence of $Np(N)$.
- It was the first result for this kind of model when $Np(N) \to c < \infty$. A few months later there is a result of Kavita and others (Arxiv-18).

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Due time limit we will discuss the case when $Np(N) \to c < \infty$.

$Np(N) \to c < \infty$

The sequence $(G_N)_{N\geq 1}$ "locally looks like" the random GW(c) tree $(\mathcal{T}, o)$ in the sense that for every reference rooted graph (H, p)

$$\lim_{N\to\infty} \frac{1}{N}\#\left\{1 \leq i \leq N : (G_N, i)_R = (H, p)_R\right\} \to \mathbb{P}\left((\mathcal{T}, o)_R = (H, p)_R\right).$$

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We can guess (heuristic):

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If we have the following (in a "good" way):

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Then we can imagine that

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\theta_i^{G_N}} \to \text{Law of } \theta_o^{\mathcal{T}}.$$

Main theorems

New object limit: We can solve for a.e. realization of $(\mathcal{T}, o)$ (with uniqueness) the possible infinite system of SDEs obtaining the process $\left(\theta_v^{\mathcal{T}}\right)_{v \in \mathcal{T}}$.

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Propagation of chaos: Let $(\mathcal{T}_1, o_1), \cdots, (\mathcal{T}_k, o_k)$ i.i.d. GW(c) trees. For bounded Lipschitz functions $f_1, \cdots, f_k$

$$\lim_{N \to \infty} \frac{1}{N^k} \sum_{u_1, \cdots, u_k = 1}^{N} \mathbb{E}\left[\prod_{i=1}^{k} f_i(\theta_{u_i}^{G_N})\right] \to \prod_{i=1}^{k} \mathbb{E}\left[f_i(\theta_{o_i}^{\mathcal{T}_i})\right].$$

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$$\frac{1}{N}\sum_{i=1}^{N} h\big(\theta_i^{G_N}\big) \asymp \frac{1}{N}\sum_{i=1}^{N} h\big(\theta_i^{(G_N,i)_R}\big) \asymp \mathbb{E}\left[h(\theta_o^{(\mathcal{T},o)_R})\right] \asymp \mathbb{E}\left[h(\theta_o^{\mathcal{T}})\right].$$

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Using these intermediate steps we can use classical concentration inequalities for functions of i.i.d. Brownian motions.

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$$\theta_i^{(G_N,i)_R} \asymp \theta_i^{(G_N,i)_\infty} = \theta_i^{G_N} \text{ and } \theta_o^{(\mathcal{T},o)_R} \asymp \theta_o^{(\mathcal{T},o)_\infty} = \theta_o^{\mathcal{T}}.$$

For example, to prove SLLN we create intermediate steps

$$\frac{1}{N}\sum_{i=1}^{N} h\left(\theta_i^{G_N}\right) \asymp \frac{1}{N}\sum_{i=1}^{N} h\left(\theta_i^{(G_N,i)_R}\right) \asymp \mathbb{E}\left[h(\theta_o^{(\mathcal{T},o)_R})\right] \asymp \mathbb{E}\left[h(\theta_o^{\mathcal{T}})\right].$$

Using these intermediate steps we can use classical concentration inequalities for functions of i.i.d. Brownian motions.
Next I will tell about the "approximations."

The “approximations”

Now I will tell you how to “prove” that

$$\theta_i^{(G_N,i)_R} \asymp \theta_i^{(G_N,i)_\infty}$$

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Using a extended version of Gronwall's inequality we can show that

$$|\theta_i^{G_N}(t) - \theta_i^{(G_N,i)_R}(t)| \leq \mathbb{P}_i^{\mathrm{SRW}}\{\text{reaches } \partial(G_N, i)_R) \text{ at time } t\}$$

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Using a bound for the SRW on G_N we conclude that

$$|\theta_i^{G_N}(t) - \theta_i^{(G_N,i)_R}(t)| \leq |\partial(G_N,i)_R| \exp\left(-\frac{R}{T}\log\frac{R}{T}\right).$$

Thanks!

References I

[CDG18] F. Coppini, H. Dietert, and G. Giacomin. A Law of Large Numbers and Large Deviations for interacting diffusions on Erd\H{o}s-R\'enyi graphs. *ArXiv e-prints*, July 2018.

[Med18] G. S. Medvedev. The continuum limit of the Kuramoto model on sparse directed graphs. *ArXiv e-prints*, February 2018.

[ORS18] Roberto I. Oliveira, Guilherme H. Reis, and Lucas M. Stolerman. Interacting diffusions on sparse graphs: hydrodynamics from local weak limits. *arXiv e-prints*, page arXiv:1812.11924, Dec 2018.

References II

[RO18] G. H. Reis and R. I. Oliveira. Interacting diffusions on random graphs with diverging degrees: hydrodynamics and large deviations. *ArXiv e-prints*, July 2018.