Abstract polymer gas.
A simple inductive proof of the Fernández-Procacci criterion

Paula Mendes Soares Fialho

Universidade Federal de Minas Gerais

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Notation

The abstract polymer gas is defined by a triple $(\mathcal{P}, \mathbf{w}, W)$:

- $\mathcal{P}$ is a countable set, its elements are called polymers;
- $\mathbf{w} = \{w_x\}_{x \in \mathcal{P}}$; $w_x \in \mathbb{C}$ is the activity of the polymer x;
- $W : \mathcal{P} \times \mathcal{P} \to \{0, 1\}$ is a function, called Boltzmann factor, such that $W(x, x) = 0$ for all $x \in \mathcal{P}$ and $W(x, y) = W(y, x)$ for all $\{x, y\} \subset \mathcal{P}$.

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We can associate a simple graph $\mathcal{G} = (\mathcal{P}, \mathcal{E})$ such that

- $\mathcal{E} = \{\{x, y\} \subset \mathcal{P}; W(x, y) = 0\}$;
- $\Gamma^*_{\mathcal{G}}(x) = \{y \in \mathcal{P};\ W(x, y) = 0\}$ is the neighborhood of a vertex x of $\mathcal{G}$;
- $I(\mathcal{G})$ is the set formed by all finite independent sets of $\mathcal{G}$.

The grand canonical partition function

Let $\Lambda \subset \mathcal{P}$ be a finite collection of polymers $\Lambda \subset \mathcal{P}$

$$Z_\Lambda(\mathbf{w}) = \sum_{\substack{S \subseteq \Lambda \\ S \in I(\mathcal{G})}} \prod_{x \in S} w_x .$$

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Problem

Find radii $\mathbf{R} = \{R_x\}_{x \in \mathcal{P}}$, *with* $R_x \geq 0$ *for all* $x \in \mathcal{P}$, *such that, for all finite* $\Lambda \subset \mathcal{P}$, *we have that*

$$Z_\Lambda(\mathbf{w}) \neq 0$$

in the polydisc $|\mathbf{w}| \leq \mathbf{R}$ *(i.e.,* $|w_x| \leq R_x$ *for all* $x \in \mathcal{P}$*).*

Theorem (Fernández-Procacci criterion)

Let $\mu = \{\mu_x\}_{x \in \mathcal{P}}$ *be a collection of nonnegative numbers such that*

$$|w_x| \le R_x^{\mathrm{FP}} \equiv \frac{\mu_x}{\varphi_x^*(\mu)} \qquad \forall x \in \mathcal{P},$$

then, for all finite $\Lambda \subset \mathcal{P}$,

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where

$$\varphi_x^*(\mu) = \sum_{\substack{S \subseteq \Gamma_{\mathcal{G}}^*(x) \\ S \in I(\mathcal{G})}} \prod_{y \in S} \mu_y .$$

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$$\varphi_x^*(\mu) = \sum_{\substack{S\subseteq\Gamma_{\mathcal{G}}^*(x)\\ S\in I(\mathcal{G})}} \prod_{y\in S} \mu_y .$$

Observe $\varphi_x^*(\mu) \equiv Z_{\Gamma_{\mathcal{G}}^*(x)}(\mu)$.

Properties of the partition function

1. *Fundamental identity:* let $\Lambda \subset \mathcal{P}$ be a finite set and $x \in \Lambda$, then

$$Z_\Lambda(\mathbf{w}) = Z_{\Lambda\setminus\{x\}}(\mathbf{w}) + w_x Z_{\Lambda\setminus\Gamma^*_{\mathcal{G}}(x)}(\mathbf{w}).$$

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2. *Log-subadditivity:* let $S, T \subseteq \mathcal{P}$ and let $\mu = \{\mu_x\}_{x\in\mathcal{P}}$ with $\mu_x \geq 0$ for all $x \in \mathcal{P}$, then

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3. Let $\mathbf{p} = \{p_x\}_{x\in\mathcal{P}}$ be a collection of nonnegative numbers. Then

$$Z_\Lambda(-\mathbf{p}) > 0 \qquad \Longleftrightarrow \qquad Z_\Lambda(\mathbf{w}) \neq 0 \;\; \forall \mathbf{w} : \; |\mathbf{w}| \leq \mathbf{p}.$$

Proposition

Let $\mu \equiv \{\mu_x\}_{x \in \mathcal{P}}$ *and* $\mathbf{p} \equiv \{p_x\}_{x \in \mathcal{P}}$ *be nonnegative numbers such that*

$$p_x \leq R_x^{\mathrm{FP}} = \frac{\mu_x}{Z_{\Gamma^*_{\mathcal{G}}(x)}(\mu)}, \quad \forall x \in \mathcal{P}.$$

Given $\Lambda \subset \mathcal{P}$ *be a finite set, let* $S \subseteq \Lambda$ *and let* $S^c = \Lambda \setminus S$*. Then*

$$\frac{Q_S(\mathbf{p})}{Q_{S \setminus \{x\}}(\mathbf{p})} \geq \frac{Z_{S^c}(\mu)}{Z_{(S \setminus \{x\})^c}(\mu)}, \quad \text{for all } x \in S,$$

where

$$Q_S(\mathbf{p}) = Z_S(-\mathbf{p}).$$

Sketch of the proof:

Let us use induction on $|S|$. For $S = \{x\}$, observe that

$$\frac{Q_{\{x\}}(\mathbf{p})}{Q_{\emptyset}(\mathbf{p})} = \frac{1 - p_x}{1} = 1 - p_x.$$

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Therefore,

$$\frac{Q_{\{x\}}(\mathbf{p})}{Q_{\emptyset}(\mathbf{p})} \ge \frac{Z_{\Lambda\setminus\{x\}}(\mu)}{Z_{\Lambda}(\mu)}.$$

Now, suppose that for any set $T \subset \Lambda$ with cardinality less than n and for any $x \in T$ we have that

$$\frac{Q_{T}(\mathbf{p})}{Q_{T\setminus\{x\}}(\mathbf{p})} \ge \frac{Z_{T^c}(\mu)}{Z_{(T\setminus\{x\})^c}(\mu)}.$$

Sketch of the proof(cont.)

Now we will show that this also holds for any set $S \subset \Lambda$ such that $|S| = n$ and for any polymer $x \in S$. Observe that

$$\frac{Q_S(\mathbf{p})}{Q_{S\setminus\{x\}}(\mathbf{p})} = \frac{Q_{S\setminus\{x\}}(\mathbf{p}) - p_x Q_{S\setminus\Gamma^*_{\mathcal{G}}(x)}(\mathbf{p})}{Q_{S\setminus\{x\}}(\mathbf{p})} = 1 - p_x \frac{Q_{S\setminus\Gamma^*_{\mathcal{G}}(x)}(\mathbf{p})}{Q_{S\setminus\{x\}}(\mathbf{p})}.$$

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Suppose that $|\Gamma^*_{\mathcal{G}}(x) \cap S| = m+1$ and let us write $\Gamma^*_{\mathcal{G}}(x) \cap S = \{x, x_1, \ldots, x_m\}$, then

$$\frac{Q_{S\setminus\{x\}}(\mathbf{p})}{Q_{S\setminus\Gamma^*_{\mathcal{G}}(x)}(\mathbf{p})} = \prod_{i=1}^{m} \frac{Q_{S\setminus\{x\cup x_1\cup\cdots\cup x_{i-1}\}}(\mathbf{p})}{Q_{S\setminus\{x\cup x_1\cup\cdots\cup x_i\}}(\mathbf{p})}$$

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Then,

$$\frac{Q_S(\mathbf{p})}{Q_{S\setminus\{x\}}(\mathbf{p})} = 1 - p_x \frac{Q_{S\setminus\Gamma^*_{\mathcal{G}}(x)}(\mathbf{p})}{Q_{S\setminus\{x\}}(\mathbf{p})} \geq \frac{Z_{S^c}(\mu)}{Z_{(S\setminus\{x\})^c}(\mu)}. \qquad \square$$

Finalizing the proof of the Fernández-Procacci criterion

Let $\Lambda \subset \mathcal{P}$ be finite and $S \subseteq \Lambda$. Suppose that $S = \{x_1, \cdots, x_n\}$, then

$$Q_S(\mathbf{p}) = \frac{Q_S(\mathbf{p})}{Q_\emptyset(\mathbf{p})} = \prod_{i=1}^{n} \frac{Q_{\{x_1 \cup \cdots \cup x_i\}}(\mathbf{p})}{Q_{\{x_1 \cup \cdots \cup x_{i-1}\}}(\mathbf{p})}.$$

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Supposing that $\mathbf{p} \leq \mathbf{R}^{\mathrm{FP}}$, i.e considering the hypothesis of Fernández-Procacci criterion valid, the proposition holds and we have that

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Therefore, we have shown that

$$Z_\Lambda(-\mathbf{p}) = Q_\Lambda(\mathbf{p}) > 0$$

for all $\mathbf{p} \leq \mathbf{R}^{\mathrm{FP}}$.

THANK YOU FOR YOUR ATTENTION!