A variational formula for functionals of fBM and applications to LDPs

André de Oliveira Gomes

IMECC-UNICAMP, Campinas SP, Brasil

Escola Brasileira de Probabilidade XXIII-2019
São Carlos, SP Brazil

joint work with Pedro Catuogno (IMECC-UNICAMP)

26 July 2019

Overview

1. The baby problem.
2. Features of the fBM.
3. A Quick tour on the weak convergence approach to LDT.
4. A variational formula for bounded measurable functionals of fBM.
5. A sufficient condition for LDPs of SDEs driven by fBM.
7. Donsker-Varadhan LDPs.

The starting point- the baby problem

We would like to understand the deviations in an exponentially small scale, when $\varepsilon \to 0$, of the solutions of the following SDEs

$$\begin{cases} dZ_t^{\varepsilon,z} & = V_0(Z_t^{\varepsilon,z})dt + \varepsilon \sum_{i=1}^{d} V_i(Z_t^{\varepsilon,z}) \circ dX_t^i, \\ Z_0^{\varepsilon,z} & = z, \end{cases} \tag{1}$$

where $(X_t)_{t\geq 0} := (X_t^1, \ldots, X_t^d)_{t\geq 0}$ is a d-dimensional Gaussian process and $V_0, \ldots, V_d$ is a collection of smooth vector fields: $\mathbb{R}^n \longrightarrow \mathbb{R}$.

The fractional Brownian Motion

The driving signal $(X_t)_{t\geq 0}$ is a fractional Brownian motion (fBM) with Hurst parameter $H \in (0,1)$, that is,

$X = B^H$, where B^H is a centered Gaussian process with covariance given by

$$R_H(s,t) = \mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big)$$

Remark

If $H = \frac{1}{2}$ the process B^H is a standard Brownian motion.

Features of the fBM

1. **Self-similarity:** For any $a > 0$, one has $\{B^H(at) \mid t \geq 0\} =^d \{a^H B^H(t) \mid t \geq 0\}$. It results from the structure of the covariance function.
2. **Stationary increments:** $\{B^H(t+h) - B^H(h)\} =^d \{B^H(t)\}$, for every $h > 0$.
3. **Independent increments:no!!** fBMs have independent increments iff $H = \frac{1}{2}$ and in this case $\mathbb{E}[B_t^H B_s^H] = t \wedge s$. When $H \neq \frac{1}{2}$ the increments are not independent. When $H > \frac{1}{2}$ the increments are positively correlated; if $H < \frac{1}{2}$ they are negatively correlated.
4. **Long range dependence:** Let $(X_t)_{t \geq 0}$ be an H-self similar process with stationary increments and non degenerate for all $t \geq 0$ with $\mathbb{E}[|X_1|^2] < \infty$. Write $\xi_n = X_{n+1} - X_n$ and $r(n) = \mathbb{E}[\xi(0)\xi(n)]$, for all $n \geq 0$. For $\frac{1}{2} < H < 1$ we have $\sum_n |r_n| = \infty$ and this property is called *long range dependence*.

Features of the fBM

5 **Markovian pp:** A Gaussian process with covariance R is Markovian iif

$$R(s,u) = \frac{R(s,t)R(s,u)}{R(t,t)}, \quad s \leq t \leq u.$$

The fBM is Markovian iif $H = \frac{1}{2}$.

6. β-**Hölder continuity:** fBM admits a modification which is Hölder continuous of order β iif $\beta \in (0, H)$. The value of the Hurst parameter decides the regularity of the sample paths.
7. **Differentiability:** fBM is a.s. nowhere differentiable.
8. **p-variation:** fBM has bounded p-variation when $p > \frac{1}{H}$ and unbounded p-variation when $p < \frac{1}{H}$.
9. **It is not a semimartingale.** If $B_t^H = A_t^H + M_t^H$ for all $t \geq 0$, by Doob-Meyer, when $H < \frac{1}{2}$ we have $[M^H]_t = \infty$ and $|A_t^H|_{TV} = \infty$ if $H > \frac{1}{2}$. Therefore no stochastic calculus. NO ITO!!!

Features of the fBM

10. How to define $\int_0^t u_s dB_s^H$?
    i) When $H > \frac{1}{2}$ one uses Young's integral.
    ii) When $H \in \left(\frac{1}{3}, \frac{1}{2}\right)$ one uses RPtheory (Coutin, Hairer, Baudoin, Gubinelli...)
    iii) Nualart's antecipative calculus via the divergence operator (Skorohod integral)

We choose Rough Paths theory.

LDP: the weak convergence approach

Let $(X^\varepsilon)_{\varepsilon>0}$ be a family of r.vs defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a complete separable metric space E (Polish).

- A function $I : E \longrightarrow [0, \infty]$ is called a good rate function if I is lower semicontinuous and if the sublevel sets $\{x \in E \mid I(x) \leq c\}$ are compact $c \geq 0$.
- The family $(X^\varepsilon)_{\varepsilon>0}$ is said to satisfy a large deviations principle on E with the good rate function I if

$$\limsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}(X^\varepsilon \in F) \leq -\inf_{x\in F} I(x)$$
$$\liminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}(X^\varepsilon \in G) \geq -\inf_{x\in G} I(x),$$

for every $F \in \mathcal{B}(E)$ closed and $G \in \mathcal{B}(E)$ open.

LDPs via the weak convergence approach

- Laplace's method: for any $h \in C_b([0,1])$ one has

$$\lim_{n\to\infty} \frac{1}{n} \ln \int_0^1 e^{-h(x)} dx = - \min_{x\in[0,1]} h(x).$$

- $(X^\varepsilon)_{\varepsilon>0}$ a family of E-valued r.vs. is said to satisfy the Laplace-Varadhan principle with the good rate function I if

$$\limsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{E}\left[e^{-\frac{1}{\varepsilon}h(X^\varepsilon)}\right] \leq - \inf_{x\in E}\{I(x) + h(x)\},$$

$$\liminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{E}\left[e^{-\frac{1}{\varepsilon}h(X^\varepsilon)}\right] \geq - \inf_{x\in E}\{h(x) + I(x)\},$$

for every $h \in C_b(E)$.

- Since E **is Polish** LDP $\Leftrightarrow$ LVP.

The relative entropy

- Let $\mathcal{P}(E)$ denote the set of probability measures defined on $(E, \mathcal{E})$. Given $\mu \in \mathcal{P}(E)$ we define $R(.||\mu) : \mathcal{P}(E) \longrightarrow [0, \infty]$ given by

$$R(\nu||\mu) := \begin{cases} \int_E \ln \frac{d\nu}{d\mu}(x)\nu(dx), & \text{if } \nu \ll \mu \text{ and } \ln \frac{d\nu}{d\mu} \in L^1(\mu) \\ \infty, & \text{otherwise.} \end{cases}$$

- **Variational representation of Laplace functionals:** Let $h \in M_b(E)$. Let $\mu \in \mathcal{P}(E)$. Then

$$-\ln \int_E e^{-h(z)}\mu(dz) = \inf_{\nu \in \mathcal{P}(E)} \left\{ R(\nu||\mu) + \int_E h(z)\nu(dz) \right\}$$

  and let $\nu_0 \in \mathcal{P}(E)$ such that $\nu_0 \ll \mu$ and

$$\frac{d\nu_0}{d\mu} = \frac{e^{-h}}{\int_E e^{-h} d\mu}.$$

  Then the infimum above is attained uniquely at ν_0.

Donsker-Varadhan representation

- Let E be a Polish space and μ and ν in $\mathcal{P}(E)$. One has the representation

$$R(\nu||\mu) = \sup_{f \in C_b(E)} \left\{ \int_e f d\nu - \ln \int_E e^f d\mu \right\}$$
$$= \sup_{\phi \in M_b(E)} \left\{ \int_e \phi d\nu - \ln \int_E e^\phi d\mu \right\}.$$

- Laplace functionals and Relative entropies are convex conjugates in the duality of the Fenchel-Legendre transform.

Fractional calculus associated to fBM

- Given $H \in (0,1)$ let $\mathcal{H}^H$ be the reproducing kernel Hilbert space associated, which consists on the functions $h : [0, T] \longrightarrow \mathbb{R}^d$ such that $\dot{h} \in L^2$ that have the representation

$$h(t) = \int_0^t K_H(t,s)\dot{h}(s)ds,$$

where K_H is the kernel defined by

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\left(\frac{1}{2} - H\right)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right)du,$$

for some $c_H > 0$.

- The scalar product in $\mathcal{H}^H$ is given by

$$\langle h_1, h_2\rangle_{\mathcal{H}^H} = \langle \dot{h}_1, \dot{h}_2\rangle_{L^2}$$

A bit of Gaussian analysis

- For every $t \in [0, T]$ we denote $\mathcal{F}_t^{B^H}$ the σ-field generated by the random variables $B_s^H, s \in [0, t]$ and the $\mathbb{P}$-null sets.
- We denote $\mathcal{E}$ the set of step functions on $[0, T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ wrt to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} := R_H(t, s).$$

- **The map $\mathbf{1}_{[0,t]} \mapsto B_t^H$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B^H)$ associated with B_H.** We will denote this isometry by $\varphi \mapsto B^H(\varphi)$.
- The covariance kernel can be written as

$$R_H(t, s) = \int_0^{t \wedge s} K_H(t, r) K_H(s, r) dr.$$

A bit of Gaussian analysis

- Consider the linear operator $K_H^* : \mathcal{E} \longrightarrow L^2[0,T]$ given by

$$(K_H^* \varphi)(s) = K_H(T,s)\varphi(s) + \int_s^T (\varphi(r) - \varphi(s)) \frac{\partial K_H}{\partial r}(r,s) dr.$$

- For any pair of step functions $\varphi, \psi \in \mathcal{E}$ we have

$$\langle K_H^* \varphi, K_H^* \psi \rangle_{L^2[0,T]} = \langle \varphi, \psi \rangle_{\mathcal{H}}.$$

- As a consequence **the operator K_H^* provides an isometry between $\mathcal{H}$ and $L^2[0,T]$.** Hence the process $W = (W_t)_{t \in [0,T]}$ defined by

$$W_t = B^H((K_H^*)^{-1} \mathbf{1}_{[0,t]})$$

is a **Wiener process wrt to** $\mathcal{F}^{B^H}$ and the process B^H has an integral representation of the form

$$B_t^H = \int_0^t K_H(t,s) dW_s,$$

since $(K_H^* \mathbf{1}_{[0,t]})(s) = K_H(t,s)$.

Girsanov's transform

- Given an $\mathcal{F}^{B^H}$-an adapted process $(u_t)_{t\in[0,T]}$ and consider the transformation

$$\tilde{B}_t^H = B_t^H + \int_0^t u_s ds.$$

- We can write

$$\begin{aligned}\tilde{B}_t^H &= B_t^H + \int_0^t u_s ds = \int_0^t K_H(t,s)dW_s + \int_0^t u_s ds \\ &= \int_0^t K_H(t,s)d\tilde{W}_s,\end{aligned}$$

where

$$\tilde{W}_t = W_t + \int_0^t \left(K_H^{-1}\Big(\int_0^{\cdot} u_s ds\Big)(r)\right)dr$$

Girsanov transform

Theorem

Consider the shifted process $\tilde{B}^H_\cdot$ defined by the process $(u_s)_{s\in[0,T]}$ with integrable paths. Assume that

- *It holds that $\int_0^\cdot u_s ds \in I_{0+}^{H+\frac{1}{2}} L^2[0,T]$ $\mathbb{P}$-a.s.*
- $\mathbb{E}[\mathcal{E}_T] = 1$ *where*

$$\mathcal{E}_T = \exp\left\{ -\int_0^T \left(K_H^{-1} \int_0^\cdot u_s ds\right)(s) dW_s - \frac{1}{2}\int_0^T \left(K_H^{-1}\int_0^\cdot u_s ds\right)^2(s)\right\}$$

Then the shifted process $\tilde{B}^H$ is an $\mathcal{F}^{B^H}$-fBM with Hurst parameter H under the new probability $\bar{\mathbb{P}}$ defined by $\frac{d\bar{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T$.

A variational formula for bounded measurable functionals of fBM-cf. Dupuis-Ellis's formula for BM

Theorem

For any $f \in M_b(C([0,T];\mathbb{R}^d))$

$$
\begin{aligned}
&-\ln \mathbb{E}\Big[e^{-f(B^H)}\Big] \\
&= \inf_{v\in\mathcal{A}} \mathbb{E}\Big[\frac{1}{2}\int_0^T \Big|K_H^{-1}\Big(\int_0^\cdot u_s ds\Big)(r)\Big|^2 dr + f\Big(B^H + \int_0^\cdot u_s ds\Big)\Big].
\end{aligned}
$$

where $\mathcal{A}$ *is the class of all* d*-dimensional* $\mathcal{F}_t^{B^H}$*-progressively measurable processes such that*

$$
\mathbb{E}\Big[\int_0^T \Big|K_H^{-1}\Big(\int_0^\cdot u_s ds\Big)\Big|^2\Big] < \infty.
$$

- We start to see that $-\ln \mathbb{E}[f(B^H)]$ is bounded below by the right hand side.
- Take $v \in \mathcal{A}_b$. Then Novikov's condition (a bit hard) shows that

$$M_t := \exp\Big(\int_0^t \Big(K_H^{-1} \int_0^{\cdot} u_r dr\Big)(s) dW_s - \frac{1}{2} \int_0^t \Big| \Big(K_H^{-1} \int_0^{\cdot} u_r dr\Big)(s)\Big|^2 ds \Big)$$

is a martingale wrt to $\mathcal{F}^{B^H}$.

- Define the measure

$$\mathbb{Q}(A) := \int_A M_T d\mathbb{P}, \quad A \in \mathcal{F}_T^{B^H}.$$

- Girsanov yields, considering $\tilde{B}_t^H = B_t^H - \int_0^t u_s ds$

$$R(\mathbb{Q}||\mathbb{P}) = \int \ln \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{Q}$$

$$= \int \Big(\int_0^T \Big(K_H^{-1} \int_0^{\cdot} u_r dr \Big)(s) dW_s - \frac{1}{2} \int_0^T \Big| \Big(K_H^{-1} \int_0^{\cdot} u_r dr \Big)(s) \Big|^2 ds \Big) d\mathbb{Q}$$

$$= \mathbb{E}_{\mathbb{Q}} \Big[\int_0^T \Big(K_H^{-1} \int_0^{\cdot} u_r dr \Big)(s) d\tilde{W}_s + \frac{1}{2} \int_0^T \Big| \Big(K_H^{-1} \int_0^{\cdot} u_r dr \Big)(s) \Big|^2 ds \Big]$$

$$= \mathbb{E} \Big[\frac{1}{2} \int_0^T \Big| \Big(K_H^{-1} \int_0^{\cdot} u_r dr \Big)(s) \Big|^2 ds \Big]$$

- One obtains

$$R(\mathbb{Q}||\mathbb{P}) + \int f d\mathbb{Q} = \mathbb{E}_{\mathbb{Q}} \Big[\frac{1}{2} \int_0^T \Big| \Big(K_H^{-1} \int_0^{\cdot} u_r dr \Big)(s) \Big|^2 ds + f\Big(\tilde{W}_t + \int_0^T K_H^{-1} \int_0^{\cdot} v(s) ds \Big) \Big]$$

- Change of measure techniques (Gaussian analysis) and density arguments prove

$$-\ln \mathbb{E}[e^{-f(B^H)}] \le \mathbb{E}\Big[\frac{1}{2}\int_0^T \Big|K_H^{-1}\Big(\int_0^\cdot u_s ds\Big)\Big|^2 + f\Big(B^H + \int_0^\cdot u_s ds\Big)\Big],$$

  for every $u \in \mathcal{A}$.
- The reverse inequality is harder!
- The way we solved it:
- **FRACTIONAL MARTINGALES AND CHARACTERIZATION OF THE FRACTIONAL BROWNIAN MOTION**, Hu, Nualart and Song, The Annals of Probability 2009, Vol. 37, No. 6, 2404–2430

A sufficient condition for a LDP

Hypothesis

- Let $\mathcal{G}^\varepsilon : C([0,T];\mathbb{R}^d) \longrightarrow E$ and $\mathcal{G}^0 : C([0,T];\mathbb{R}^d) \longrightarrow E$ with E a Polish space. Define $X^\varepsilon = \mathcal{G}^\varepsilon(B^H)$.
- Given $M > 0$ consider $\mathcal{A}_M$ the class of functions u such that $\int_0^T |K_H^{-1}\Big(\int_0^\cdot u_s ds\Big)|^2 \leq M$.
- We assume:
  1. The set $K_M = \{\mathcal{G}^0\Big(K_H^{-1}\Big(\int_0^\cdot v(s)ds\Big) \mid v \in \mathcal{A}_M\Big)\}$ is compact for all $M > 0$.
  2. Let $v_\varepsilon \in \mathcal{A}_M$ be any family of $\mathcal{A}_M$-r.vs such that $v_\varepsilon \Rightarrow v$ as $\varepsilon \to 0$. Then we have that $\mathcal{G}^0\Big(K_H^{-1}\Big(\int_0^\cdot v_s ds\Big)\Big)$ is an accumulation point in law of $\mathcal{G}^\varepsilon\Big(B^H + \frac{1}{\varepsilon}\int_0^\cdot v_\varepsilon(s)ds\Big)$.

Thm. A sufficient condition for a LDP

Theorem

Under the hypothesis above, the family $(X^\varepsilon)_{\varepsilon>0}$ satisfies a LDP with the good rate function

$$I(f) = \inf_v \frac{1}{2}\int_0^T \left|K_H^{-1}\Big(\int_0^r v_s ds\Big)(r)\right|^2 dr$$

where the inf *is taken on*

$$\Big\{v : K_H^{-1}\Big(\int_0^\cdot v_s ds\Big) \in L^2 : f = \mathcal{G}^0\Big(K_H^{-1}\Big(\int_0^\cdot v_s ds\Big)\Big)\Big\}.$$

A quick tour on Gubinelli's controlled rough paths theory

- Denote by $\Omega\mathcal{C}$ the set of continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ that are 0 on the diagonal and define the increment operator $\delta : \mathcal{C} \longrightarrow \Omega\mathcal{C}$ by

$$\delta A_{st} := A_t - A_s.$$

- For a continuous function $f : [0, T] \longrightarrow \mathbb{R}^n$ set

$$||f||_\infty := \sup_{t\in[0,T]} |f_t|, \quad ||f||_\gamma := \sup_{t\in[0,T]} \frac{|\delta f_{st}|}{|t-s|^\gamma}.$$

We define the norm $||f||_{\mathcal{C}^\gamma} := ||f||_\infty + ||f||_\gamma$.

Rough path

- A rough path on the interval $[0, T]$ consists of two parts, a continuous function $X : [0, T] \longrightarrow \mathbb{R}^d$ and a continuous (area process) $\mathbb{X} : [0, T]^2 \longrightarrow \mathbb{R}^{d\times d}$ where $\mathbb{X} \in (\Omega\mathcal{C})^{d\otimes d}$ satisfying the algebraic prop. for all $s \leq u \leq t$, i, j

$$\mathbb{X}^{i,j}_{st} - \mathbb{X}^{i,j}_{ut} - \mathbb{X}^{ij}_{su} = \delta X^i_{su}\delta X^j_{ut}. \tag{2}$$

  For $\mathbb{X} \in (\Omega\mathcal{C})^{d\otimes d}$ define

$$||\mathbb{X}||_{2\gamma} := \sup_{s\neq t\in[0,T]} \frac{|\mathbb{X}|_{st}}{|t-s|^{2\gamma}}.$$

- For $\gamma \in (\frac{1}{3}; \frac{1}{2}]$ we denote $\mathcal{D}^\gamma([0, T]; \mathbb{R}^d)$ the space of all rough paths consisting of those pairs $(X, \mathbb{X})$ satisfying (2) and such that

$$||(X, \mathbb{X})||_\gamma := ||X||_\gamma + ||\mathbb{X}||_{2\gamma} < \infty.$$

Geometric rough paths

- N.b. $||(X, \mathbb{X})||_\gamma$ is only a seminorm and $\mathcal{D}^\gamma$ is actually not a vector space due to the nonlinear constraint (2).
- For every smooth function $X : [0, T] \longrightarrow \mathbb{R}^d$ there exists a canonical representative in $\mathcal{D}^\gamma$ by choosing

$$\mathbb{X}_{st} = \int_t^s \delta X_{sr} \otimes dX_r .$$

  We denote $\mathcal{D}_g^\gamma$ the closure of the set of smooth functions in $\mathcal{D}^\gamma$. The space $\mathcal{D}_g^\gamma$ is a Polish space (always nice!).
- the fBM lifts to a geometric rough path in $\mathcal{D}_g^\gamma$ with $\frac{1}{3} < \gamma < H$.

Controlled RPs

- Let $\mathbf{X} := (X, \mathbb{X}) \in \mathcal{D}^\gamma([0,T];\mathbb{R}^d)$ for some $\gamma \in (\frac{1}{3}, \frac{1}{2}]$. A pair (Z, Z') is controlled by $\mathbf{X}$ if $Z \in \mathcal{C}^\gamma([0,T];\mathbb{R}^n)$, $Z' in \mathcal{C}^\gamma([0,T];\mathbb{R}^{n\times d})$ and the remainder $R^Z \in \Omega\mathcal{C}$ defined by

$$\delta Z_{st} = Z'_s \delta X_{st} + R^Z_{st},$$

satisfies $||R^Z||_{2\gamma} < \infty$.

- Denote by C^γ_X the set of paths controlled by X endowed with the norm

$$||(Z, Z')||_{X,\gamma} := |Z(0)| + ||Z'||_{\mathcal{C}^\gamma} + ||R^Z||_{2\gamma}.$$

- We can define for $(Z, Z') \in C^\gamma_X$ a rough integral by the Riemann sums

$$\int_0^T Z_t \otimes dX_t := \lim_{|P|\to 0} \sum_{[s,t]\in P} (Z_s \otimes \delta X_{st} + Z'_s \mathbb{X}_{st})$$

Continuity of the integral wrt to the integrand

- Let $(X, \mathbb{X}) \in \mathcal{D}^\gamma([0,T];\mathbb{R}^d)$ for some $\gamma > \frac{1}{3}$. and $(Y, Y') \in \mathcal{C}_X^\gamma$ be a controlled rough path. Then the map

$$(Y, Y') \mapsto (Z, Z') := \Big(\int_0^\cdot Y_t \otimes dX_t ; Y \Big)$$

where the integral is defined as above is continuous from $\mathcal{C}_X^\gamma$ to $\mathcal{C}_X^\gamma$ and for some $M > 0$ independent of X and Y,

$$||R^Z||_{2\gamma} \leq M(||X||_\gamma ||R^Y||_{2\gamma} + ||\mathbb{X}||_{2\gamma} ||Y'||_{\mathcal{C}^\gamma}),$$

- For $(Y, Y') \in \mathcal{C}_X^\gamma$ and $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ C^2 we define the weakly controlled rough path $(\psi(Y);\psi(Y')) \in \mathcal{C}_X^\gamma$ as

$$\psi(Y)_t = \psi(Y_t), \quad \psi(Y)'_t = D\psi(Y_t)Y'_t.$$

Solution of the DE (1)

- Let $\gamma > \frac{1}{3}$ and let $(X, \mathbb{X}) \in \mathcal{D}^\gamma$. Then $Z \in \mathcal{C}^\gamma$ is a solution to (1) if $(Z, Z') = (Z, V(Z)) \in \mathcal{C}_X^\gamma$ and the integral version of (1) holds where the composition of a controlled rough path with a nonlinear function is interpreted as before and the integral of a controlled rough path against X is defined as above.
- If $V \in C^3$ there exists a unique local solution.
- If $V \in C_b^3$ there exists a global solution (no harm in the large deviations regime).

The baby problem

- Ito-Lyons continuity theorem states that for every $\varepsilon > 0$ there exists $\mathcal{G}^\varepsilon : C([0,T];\mathbb{R}^d) \longrightarrow \mathcal{D}_g^\gamma([0,T];\mathbb{R}^d)$ with $\gamma < H$. In particular $\mathcal{D}_g^\gamma$ is Polish and good for weak compactness arguments.
- Ito-Lyons map is continuous (robust under approximations and weak compactness arguments) if we view it restricted to $\mathcal{C}^\gamma$, $\gamma < H$.
- **Skeleton equation**

$$\mathcal{G}^0(v) = \tilde{Z}_t^v = z_0 + \int_0^t V_0(\tilde{Z}_s^v)ds + \sum_{j=1}^d \int_0^t V_i(\tilde{Z}_s^v)v_s ds,$$

where the control $v \in I_{0+}^{H+\frac{1}{2}} L^2[0,T]$ which is equivalent to say

$$\int_0^T \left| K_H^{-1}\Big(\int_0^r v_s ds\Big)(r)\right|^2 dr < \infty.$$

Verifying sufficient condition for the LDP

- **Condition 1 is verified** if given $v_n, v \in I_{0+}^{H+\frac{1}{2}} L^2[0,T]$ such that

$$K_H^{-1} \int_0^{\cdot} v^n(s)ds \rightharpoonup K_H^{-1} \int_0^{\cdot} v(s)ds,$$

then $\mathcal{G}^0(v_n) \to \mathcal{G}^0(v)$.

- **Condition 2 is verified** if we prove that the family $(\tilde{Z}^\varepsilon)_{\varepsilon>0}$ is compact (through tightness arguments in $\mathcal{D}_g^\gamma$) (Polish space!)

$$d\tilde{Z}_t^\varepsilon = V_0(\tilde{Z}_t^\varepsilon)dt + \varepsilon \sum_{j=1}^{d} V_i(\tilde{Z}_t^\varepsilon)d\tilde{B}_t^H,$$

whenever $u^\varepsilon \Rightarrow u$ with u^ε, u are $\mathcal{A}_M$- random variables and

$$\tilde{B}_t^H = B_t^H + \frac{1}{\varepsilon} \int_0^{\cdot} u^\varepsilon(s)ds.$$

and therefore converging (Lyons map/Davies estimate again) to $\tilde{Z}^u$.

Conclusion

The family $(Z^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle in the geometric rough path space $\mathcal{D}_g^\gamma([0,T],\mathbb{R}^d)$ with $\frac{1}{3}<\gamma<H$ with the **good rate function** given by

$$\mathbb{I}(\varphi)=\inf_{f=\mathcal{G}^0(v)}\frac{1}{2}\int_0^T\left|K_H^{-1}\Big(\int_0^s u_r dr\Big)(s)\right|^2 ds$$

A 2d LDP-Donsker-Varadhan type: $\varepsilon = 1$!

- **Object: The occupation measure** L_t of the solution Z given by

$$L_t(A) := \frac{1}{T}\int_0^T \delta_{Z_T}(A)dsds, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

- Note that L_t is an $\mathbb{M}_1(\mathbb{E}^d)$-valued r.v, space of probability measures with the Borel σ-field $\mathcal{B}(\mathbb{R}^d)$, with the dual action between $\nu \in \mathbb{M}_1$ and $f \in \mathbb{B}_b$,

$$\nu(f) := \int f d\nu.$$

- Whenever μ is an invariant measure of Z we know that

$$L_t \Rightarrow \mu, \mathbb{P}_\mu - a.s.$$

The rate function of the 3d level LDP Donsker-Varadhan type

- The DV-2level entropy is given by

$$H(\mathbb{Q}) := \begin{cases} \mathbb{E}^{\bar{\mathbb{Q}}} R_{\mathcal{F}_t^0}(\bar{\mathbb{Q}}_{(-\infty,0]} || \mathbb{P}), & \text{if } \mathbb{Q} \in \mathbb{M}_1^s, \\ \infty & \text{otherwise} \end{cases}$$

  where

  - $\mathbb{M}_1^s$ is the space of stationary measures of $\mathbb{M}_1(\Omega)$, $\Omega = C([0,T];\mathbb{R}^d)$;
  - $\bar{\mathbb{Q}}$ is the unique stationary extension of $\mathbb{Q}$ to $\bar{\Omega} = C(\mathbb{R};\mathbb{R}^d)$;
  - $\bar{\mathbb{Q}}_{(-\infty,t]}$ is the regular conditional distribution of $\bar{\mathbb{Q}}$ knowing $\mathcal{F}_t^{-\infty}$.

- Let the **3d level entropy functional** $J : \mathbb{M}_1 \longrightarrow [0,\infty]$

$$J(\beta) := \inf\{H(\mathbb{Q}) : \mathbb{Q} \in \mathbb{M}_1^s, \mathbb{Q}_0 = \beta\}, \quad \beta \in \mathbb{M}_1(\mathbb{R}^d).$$

A 2d LDP-Donsker-Varadhan type: $\varepsilon = 1$!

- The 2d level rate entropy J is a **good rate function on $\mathbb{M}_1$** equipped with the topology τ of the convergence against bounded and Borelian functions, i.e. $[J \leq a]$ is compact for any $a \geq 0$.
- For all open sets $G \in \mathbb{M}_1$ open and F closed wrt τ

$$\liminf_{T\to\infty} \frac{1}{T} \ln \mathbb{P}_\nu(L_T \in G) \geq -\inf_G J;$$
$$\limsup_{T\to\infty} \frac{1}{T} \ln \mathbb{P}_\nu(L_T \in F) \leq -\inf_F J.$$

Comments and open doors

- The weak convergence approach for the 1d level LDPS here developed is good for vector fields with low regularity and to approach problems in infinite dimensions.
- The arguments of exponential tightness got reduced to arguments of verifying tightness of the laws. This task is free lunch from Davies estimates for rough integrals and RDEs.
- The 2d DV LDPs are obtained by a variational formulation for the occupancy measures also: they solve an infinite system of PDEs in the distributional sense: easy characterization. cf. Baudoin-Coutin.
- Next chapter: $T = T(\varepsilon)$. **FW theory+DV theory coupling.** Applications to stochastic dynamics: solving **Kramers problem without Markovianity!**

References

- S.R.S. Varadhan, **Asymptotic probabilities and differential equations**. COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XIX, 261-286 (1966)
- M. D. Donsker. S.R.S. Varadhan. **Asymptotic evaluation of certain Markov processes expectations for large times I-IV**. (1975),(1976),(1983)
- L. Coutin, Z. Quian. **Stochastic Analysis, rough paths analysis and fractional Brownian Motion**. Probab. Theory Related Fields 12 (2002)
- L. Decreusefond, A.S. Üstunel. **Stochastic analysis of the fractional Brownian motion.** Potential Anal. 10, 177-214 (1999)

Bibliography on the subject

- A. Millet, M. S-Solé. **Large deviations for rough paths of the fractional Brownian motion.** Ann Inst Poincaré Prob-Stat. vol. 41 245-271 (2006)
- C. Tudor. **Large deviations for stochastic differential equations driven by fractional Brownian motion.**
- V. Maroulas. J. Xiong. **Large deviations for optimal filtering with fractional Brownian motion.** Stoch. Proc. and their Appl. 123, 2340-2352 (2013)
- F. Baudoin. L. Coutin. **Operators associated with a stochastic differential equation driven by fractional Brownian motion.** Stoch. Proc. and their Appl. 117, 550-574 (2007)