Infinite DLR Measures and Volume-Type Phase Transitions on Countable Markov Shifts

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Outline

Countable Markov Shifts

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- Countable Markov shifts Σ_A, in general, are not locally compact.

Renewal shift

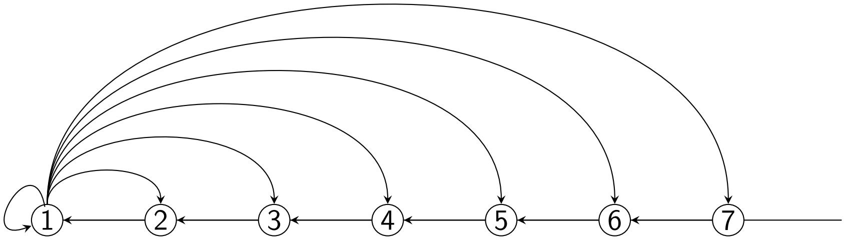

Figure: The Renewal shift Σ_A

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$\mathrm{var}_n(\phi) = \sup\{|\phi(x) - \phi(y)| : x_0 = y_0, \ldots, x_{n-1} = y_{n-1}\}$

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$\mathrm{var}_n(\phi) \leq \text{constant}.\lambda^n, 0 < \lambda < 1$ **(locally Hölder)**

$\mathrm{var}_1(\phi) = +\infty$ **is allowed.**

Ruelle operator

$\phi : \Sigma_A \to \mathbb{R}$ be a measurable potential.

Ruelle operator: For measurable function f and $x \in \Sigma_A$,

$$L_\phi(f)(x) = \sum_{\substack{y \in \Sigma_A \\ \sigma(y) = x}} e^{\phi(y)} f(y).$$

Let μ sigma-finite measure, $\lambda > 0$. **(eigenmeasures)**

$$\int L_\phi f(x) d\mu(x) = \lambda \int f(x) d\mu(x), \quad \text{for each } f \in L^1(\mu)$$

Notation: $L_\phi^*(\mu) = \lambda\mu$

Eigenmeasures from the Generalized Ruelle-Perron-Frobenius' Theorem are sigma-finite but can be infinite!

Motivation

""""""" - We should consider infinite measures on statistical mechanics, people in ergodic theory already did this..."""""

by Charles Pfister at CIRM, Marseille, in 2013.

Classical Reference: *An Introduction to Infinite Ergodic Theory, 1997*
By Jon Aaronson.

DLR Measures on the Reversal Renewal Shift

Before the infinite case...

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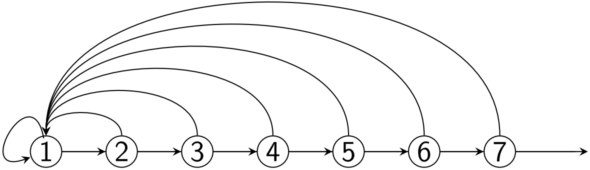

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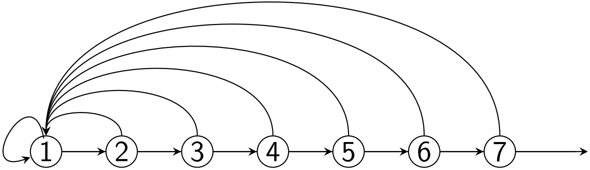

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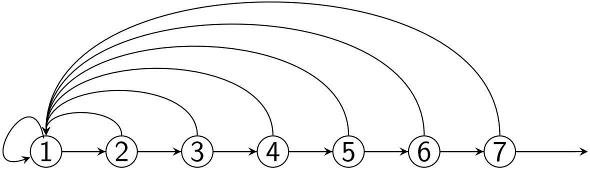

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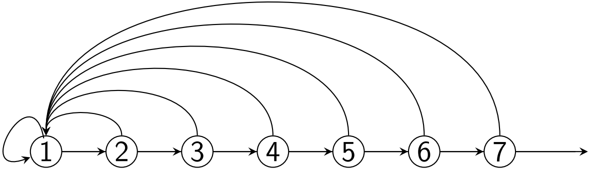

Take $z = 12345... \in \Sigma_A$ and $\nu = \delta_z$.

ν is a DLR measure... for which potential???

ν **is a DLR measure for ANY potential!!!!!!**

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Infinite DLR Measures

Definition

Let Σ_A be a Markov shift, ν be a measure on the Borel sigma algebra $\mathcal{B}$, and $\phi : \Sigma_A \to \mathbb{R}$ be a measurable potential. We say that ν is *ϕ-DLR* if, for every $n \geq 1$,

i) the restriction of ν to the sub-σ-algebra $\sigma^{-n}\mathcal{B}$ is sigma-finite,

ii) for every cylinder $[a]$ of length n, we have

$$\mathbb{E}_\nu\big(\mathbf{1}_{[a]}|\sigma^{-n}\mathcal{B}\big)(x) = \frac{e^{\phi_n(a\sigma^n x)}\mathbf{1}_{\{a\sigma^n x\in\Sigma_A\}}}{\sum\limits_{\sigma^n y=\sigma^n x} e^{\phi_n(y)}}, \qquad \nu\text{-a.e.} \tag{1}$$

Infinite DLR Measures

Proposition

$\phi : \Sigma_A \to \mathbb{R}$ be a measurable potential and ν such that $\|L_\phi \mathbf{1}\|_\infty < \infty$. If,

- *$\nu([a]) < \infty$ for each $a \in \mathbb{N}$.*
- *$L_\phi^*(\nu) = \lambda\nu$*

Then, ν is ϕ-DLR.

Phase Transitions

Buzzi-Sarig (ETDS-2003):
Let Σ_A be a topologically mixing Markov shift, if $\phi : \Sigma_A \to \mathbb{R}$ is regular enough with $\sup \phi < \infty$ and $P_G(\phi) < \infty$. Then there exists at most one equilibrium measure m and, when does exist, $m = hd\mu$ where h and μ are the eigenfunction and eigenmeasure associated to $\lambda = e^{P_G(\phi)}$.

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Theorem (Sarig - CMP - 2001)

Let Σ_A be the renewal shift and let $\phi : \Sigma_A \to \mathbb{R}$ be a weakly Hölder continuous function such that $\sup \phi < \infty$.
Then there exists $0 < \beta_c \leq \infty$ such that:

(i) *For $0 < \beta < \beta_c$, there exists $m_\beta = h_\beta \mu_\beta$ equilibrium measure for $\beta\phi$.*

(ii) *For $\beta_c < \beta$, there is no m_β equilibrium measure for $\beta\phi$.*

Volume Type Phase Transition

Theorem (RB, E.R. Beltrán, E.O. Endo, 2019+)

Let Σ_A be the renewal shift and let $\phi : \Sigma_A \to \mathbb{R}$ be a locally Hölder continuous such that $\sup \phi < \infty$. For $\beta > 0$, consider ν_β be the eigenmeasure associated to the potential $\beta\phi$. Let $\beta_c \in (0, +\infty]$ from Sarig's theorem. Then, there exists $\tilde{\beta}_c \in (0, \beta_c]$ such that:

(i) *For $0 < \beta < \tilde{\beta}_c$, ν_β is finite.*

(ii) *For $\tilde{\beta}_c < \beta < \beta_c$, ν_β is infinite.*

$$\tilde{\beta}_c = \sup \left\{ \beta \in (0, \beta_c] : \limsup_{n\to\infty} \frac{1}{n} \sum_{j=2}^{n} \phi(\gamma_j) < \frac{P_G(\beta\phi)}{\beta} \right\}$$

where $\gamma_j = \overline{(j, j-1, j-2, ..., 1)}$.

Volume Type Phase Transition

Examples:

β_c and $\tilde{\beta}_c$ can be different or equal:

i) $\phi(x) \equiv c$ ($c \in \mathbb{R}$) constant potential, then $\beta_c = \tilde{\beta}_c = +\infty$.

ii) Let $\phi(x) = x_0 - x_1$ we have $\beta_c = +\infty$ and $\tilde{\beta}_c = \log 2$.

Remark: $\log 2$ is the Gurevich's entropy of the Renewal.

Volume Type Phase Transition

Further Questions:

i) Infinite DLR measures on $\Sigma_A \subset \mathbb{N}^{\mathbb{Z}^d}$?

ii) Infinite DLR measures on classical models ?