The Geometry of Last Passage Percolation

The Geometry of Last Passage Percolation

Joint with Chris Janjigian and Timo Seppäläinen

Inviscid stochastic Burgers equation: $\partial_t u + u\,\partial_x u = \partial_x \mathcal{F}$

random forcing

Inviscid stochastic Burgers equation: $\partial_t u + u\,\partial_x u = \partial_x \mathcal{F}$

random forcing

Hamilton-Jacobi-Bellman equation (HJB): $u = \partial_x \mathcal{U}$

$$\partial_t \mathcal{U} + \frac{1}{2}(\partial_x \mathcal{U})^2 = \mathcal{F}$$

Inviscid stochastic Burgers equation: $\partial_t u + u\,\partial_x u = \partial_x \mathcal{F}$

random forcing

Hamilton-Jacobi-Bellman equation (HJB): $u = \partial_x \mathcal{U}$

$$\partial_t \mathcal{U} + \frac{1}{2}(\partial_x \mathcal{U})^2 = \mathcal{F}$$

Viscosity solution with initial condition $\mathcal{U}(t_0,x)=\mathcal{U}_0(x)$:

Lax-Oleinik: $\mathcal{U}(t,x) = \inf\limits_{\substack{\gamma:[t_0,t]\to\mathbb{R}\\ \gamma(t)=x}}$

abs. cont.

Inviscid stochastic Burgers equation: $\partial_t u + u\,\partial_x u = \partial_x \mathcal{F}$

random forcing

Hamilton-Jacobi-Bellman equation (HJB): $u = \partial_x \mathcal{U}$

$$\partial_t \mathcal{U} + \frac{1}{2}(\partial_x \mathcal{U})^2 = \mathcal{F}$$

Viscosity solution with initial condition $\mathcal{U}(t_0,x)=\mathcal{U}_0(x)$:

Lax-Oleinik: $$\mathcal{U}(t,x) = \inf_{\substack{\gamma:[t_0,t]\to\mathbb{R} \\ \gamma(t)=x}} \left\{ \mathcal{U}_0(\gamma(t_0)) + \frac{1}{2}\int_{t_0}^{t} (\gamma'(s))^2\,ds + \int_{t_0}^{t} \mathcal{F}(s,\gamma(s))\,ds \right\}$$

abs. cont.

Inviscid stochastic Burgers equation: $\partial_t u + u\,\partial_x u = \underbrace{\partial_x \mathcal{F}}$

random forcing

Hamilton-Jacobi-Bellman equation (HJB): $u = \partial_x \mathcal{U}$

$$\partial_t \mathcal{U} + \frac{1}{2}(\partial_x \mathcal{U})^2 = \mathcal{F}$$

Viscosity solution with initial condition $\mathcal{U}(t_0,x)=\mathcal{U}_0(x)$:

Lax-Oleinik: $\mathcal{U}(t,x) = \inf_{\substack{\gamma:[t_0,t]\to\mathbb{R} \\ \gamma(t)=x}} \Big\{\mathcal{U}_0(\gamma(t_0)) + \underbrace{\frac{1}{2}\int_{t_0}^{t}(\gamma'(s))^2\,ds + \int_{t_0}^{t}\mathcal{F}(s,\gamma(s))\,ds}_{\text{Action}}\Big\}$

abs. cont.

Velocity of minimizing $\gamma \to u(t,x)$ (Solutions to Burgers)

Velocity of minimizing $\gamma \to u(t,x)$ (Solutions to Burgers)

Global (bi-infinite-time) solutions: $t_0 \to -\infty$

Velocity of minimizing $\gamma \to u(t,x)$ (Solutions to Burgers)

Global (bi-infinite-time) solutions: $t_0 \to -\infty$

Related to semi-infinite minimizing paths $\gamma : (-\infty, t] \to \mathbb{R}$ s.t. $\gamma(t) = x$

Velocity of minimizing $\gamma \to u(t,x)$ (Solutions to Burgers)

Global (bi-infinite-time) solutions: $t_0 \to -\infty$

Related to semi-infinite minimizing paths $\gamma : (-\infty, t] \to \mathbb{R}$ s.t. $\gamma(t) = x$

(minimizes the action between $(t_0, \gamma(t_0))$ & (t,x) $\forall t_0 < t$)

Conserved quantity: asymptotic velocity $v = \lim_{|x|\to\infty} \frac{\mathcal{U}(t,x)}{x}$

Conserved quantity: asymptotic velocity $\quad v = \lim_{|x| \to \infty} \frac{u(t,x)}{x}$

Translates to semi-inf. minimizer having an asymptotic direction

Conserved quantity: asymptotic velocity $v = \lim_{|x| \to \infty} \frac{u(t,x)}{x}$

Translates to semi-inf. minimizer having an asymptotic direction

<u>Questions:</u> existence & uniqueness/stability of global sol. for a give v

Conserved quantity: asymptotic velocity $v = \lim_{|x|\to\infty} \frac{u(t,x)}{x}$

Translates to semi-inf. minimizer having an asymptotic direction

Questions: existence & uniqueness/stability of global sol. for a give v

Answered for certain $\mathcal{F}$'s (kick forcing): Bakhtin, Cator, Khanin '14
Bakhtin '16

Also, in compact & essentially compact space settings: Sinai et al
Khanin et al

All for a fixed value of the conserved quantity v

All for a fixed value of the conserved quantity v

Uniqueness/stability for all v?

All for a fixed value of the conserved quantity v

Uniqueness/stability for all v?

We study a space & time discrete version

All for a fixed value of the conserved quantity v

Uniqueness/stability for all v?

We study a space & time discrete version

We give a complete description for all v simultaneously

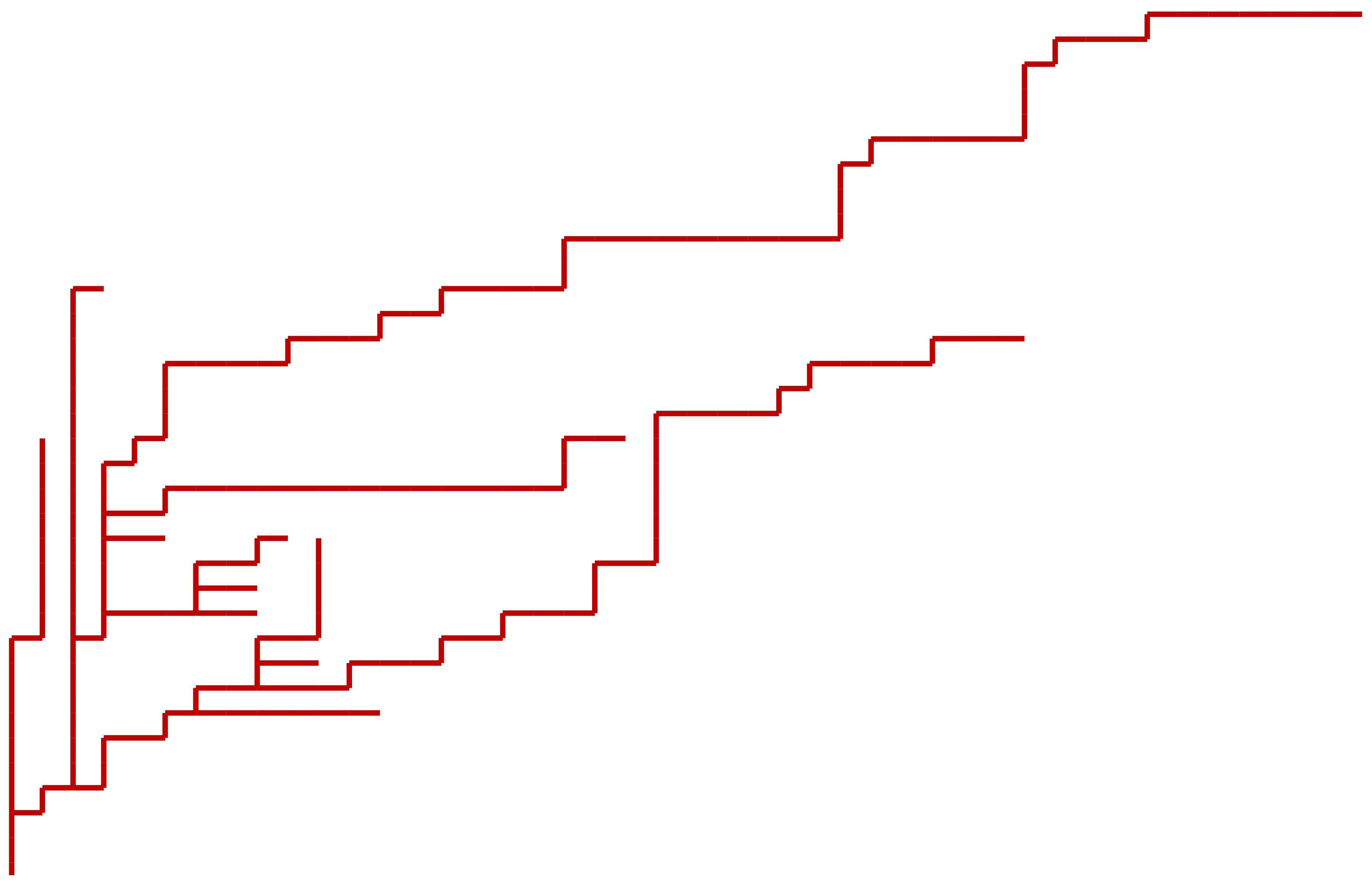
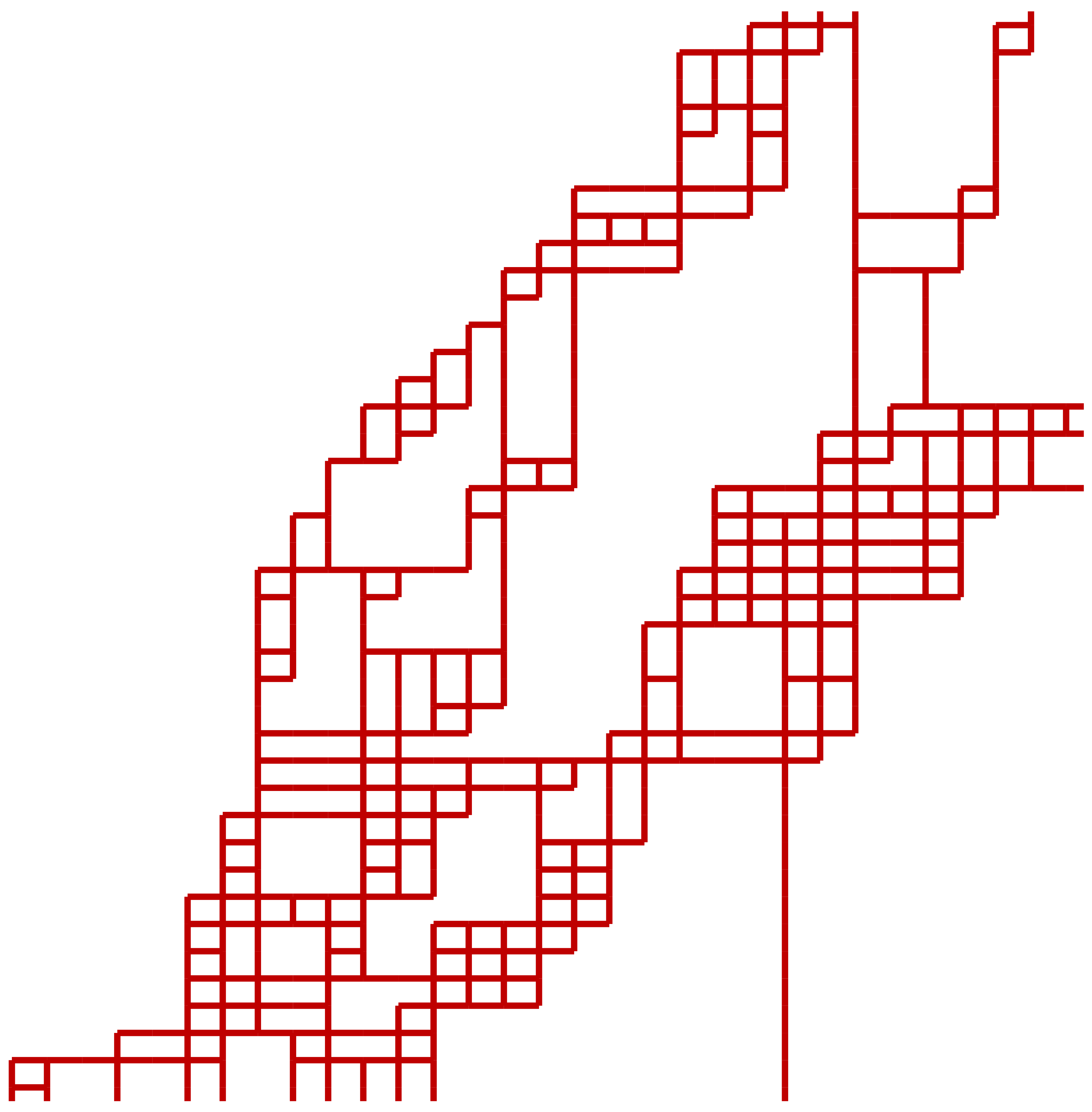
All for a fixed value of the conserved quantity v

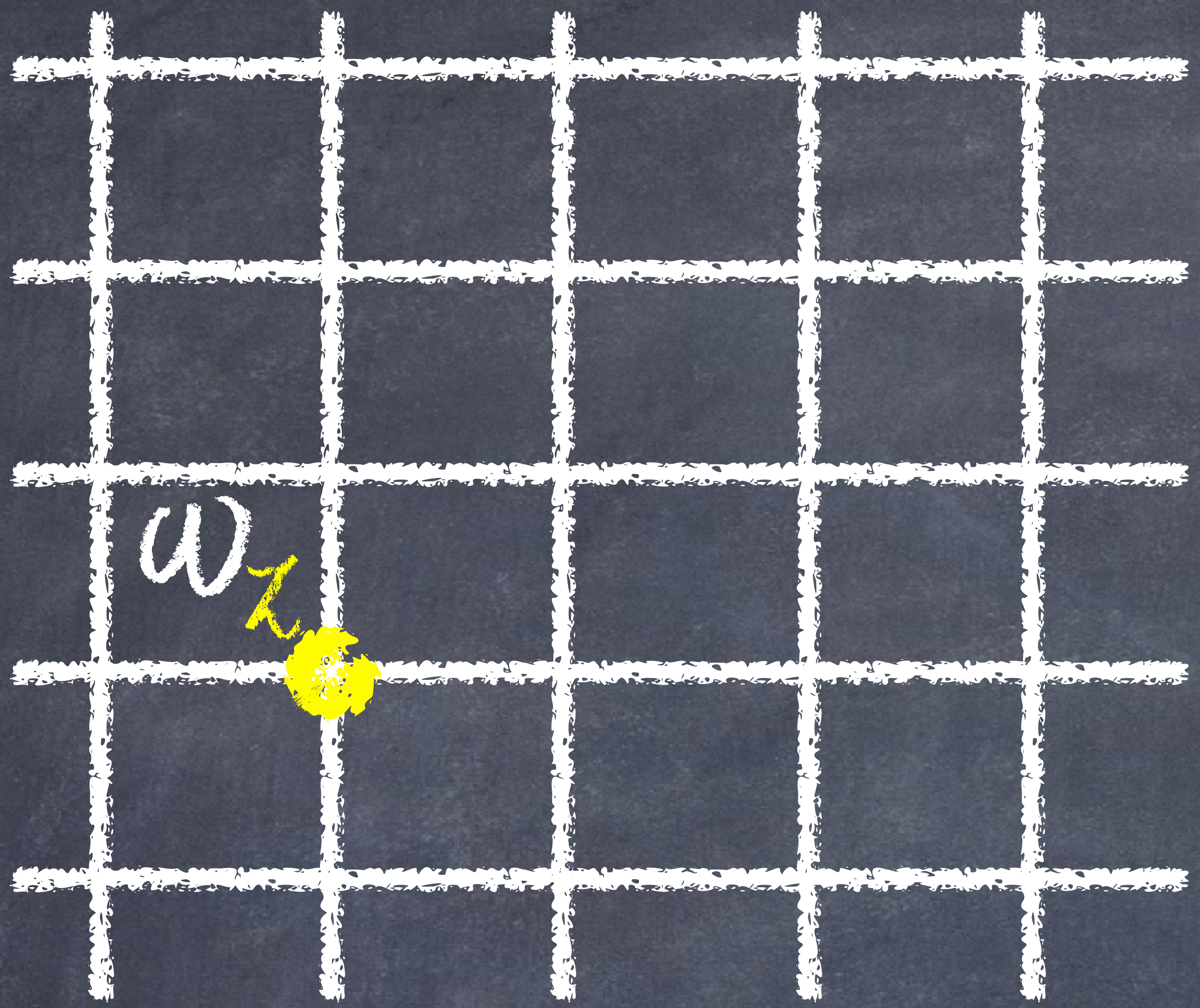
Uniqueness/stability for all v?

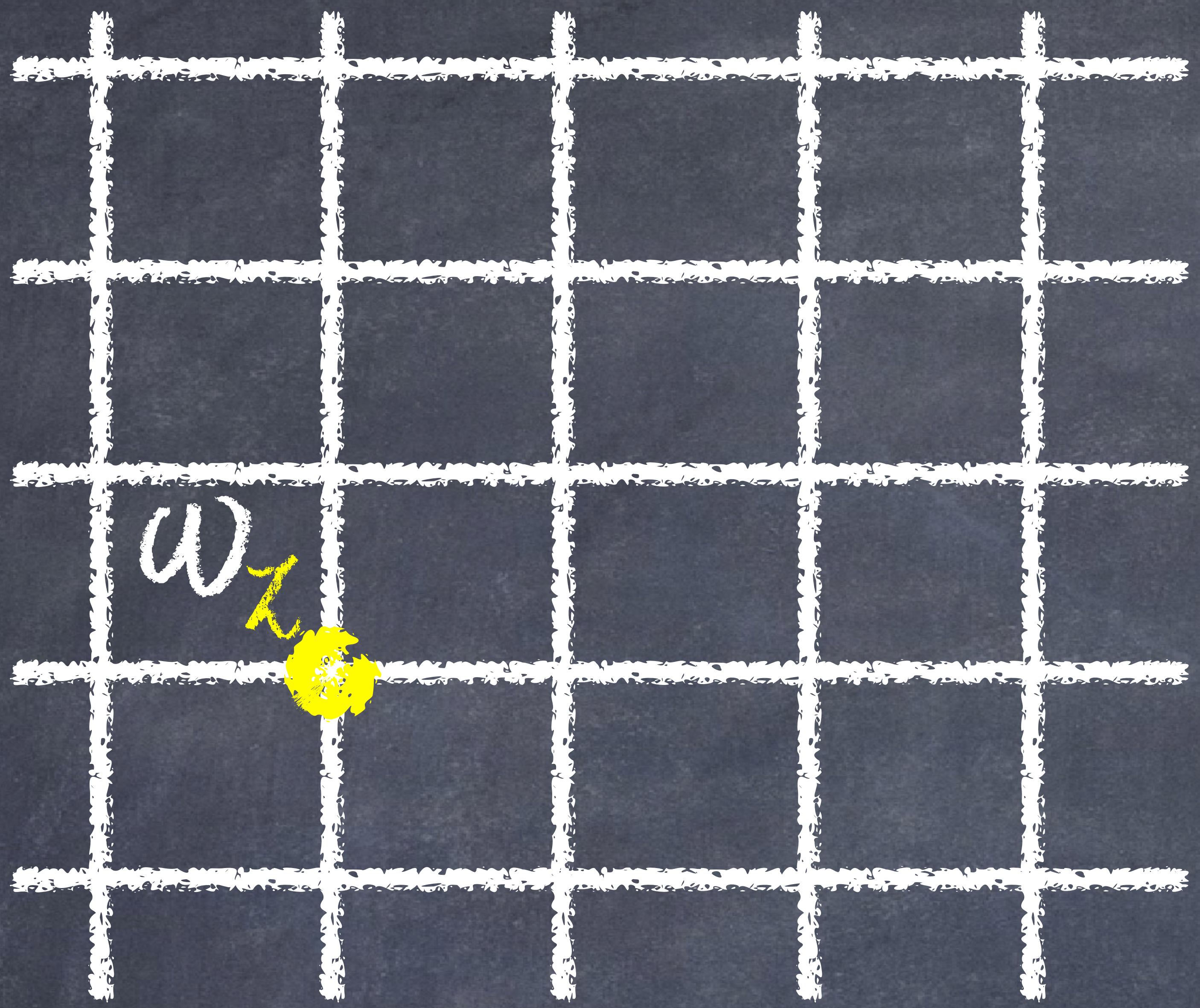
We study a space & time discrete version

We give a complete description for all v simultaneously

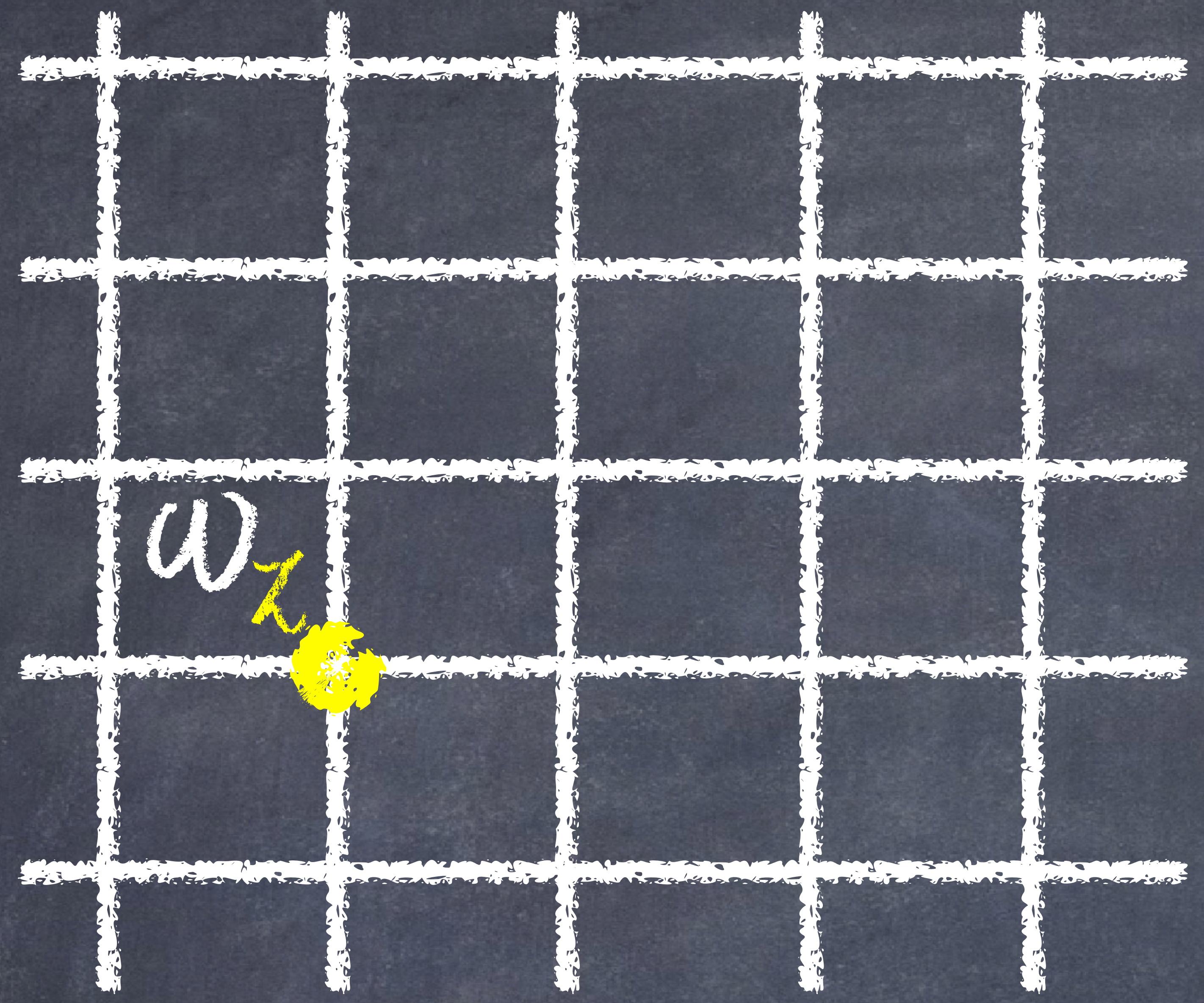
We find new instability structures for certain random v


ω_i

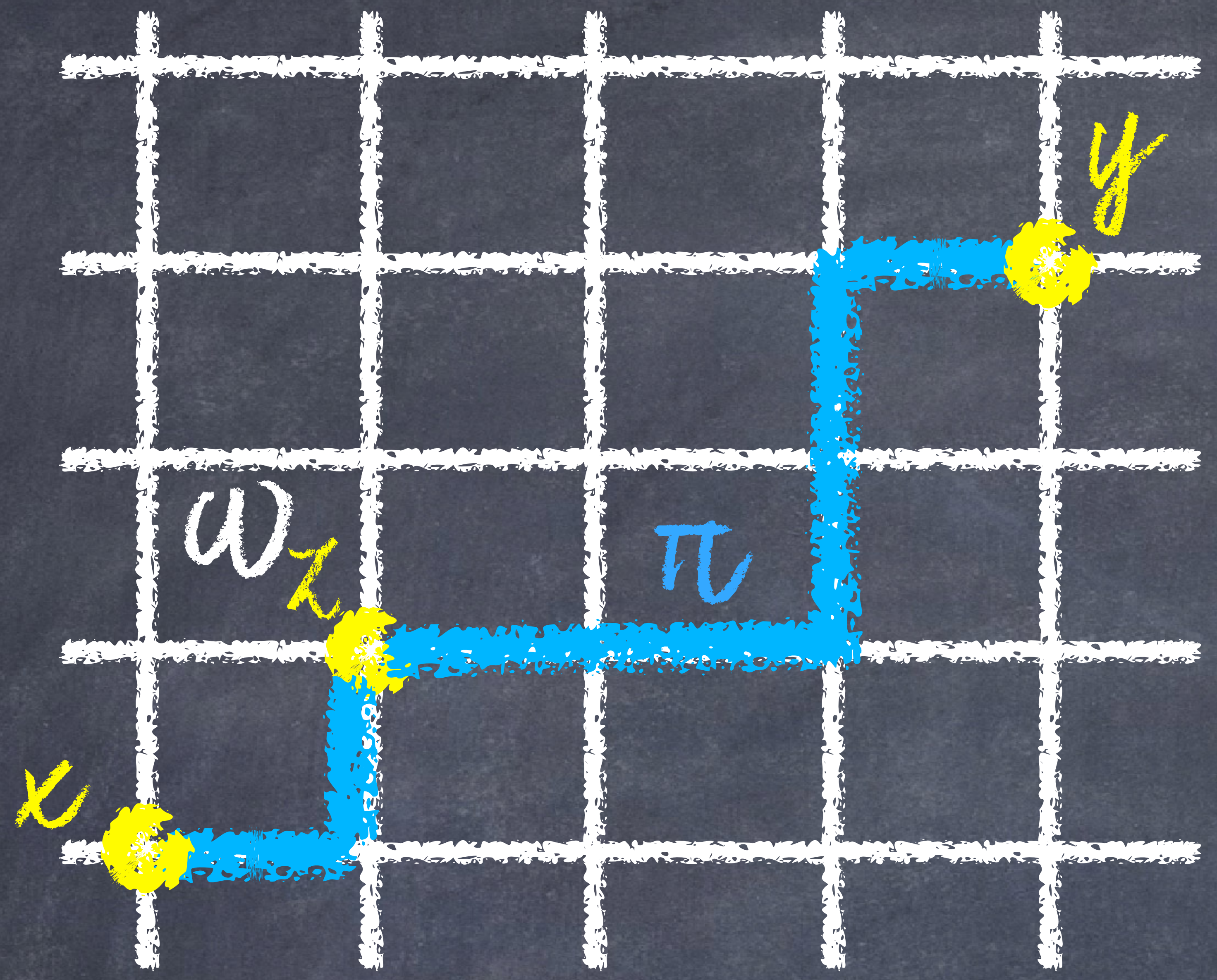


$d=2$, ω_z i.i.d., >2 moments



$d=2$, ω_z i.i.d., >2 moments

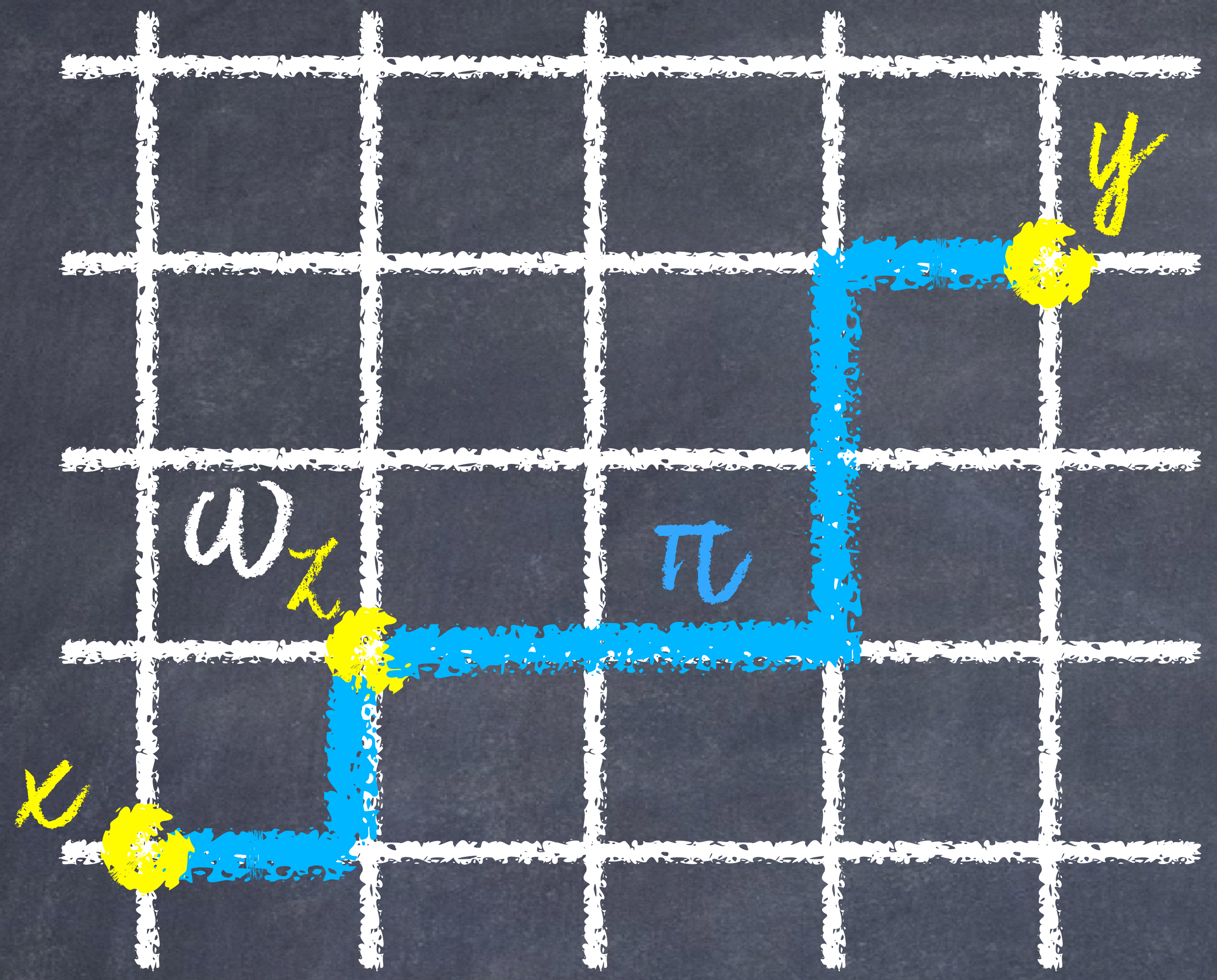
ω_z has a continuous CDF



$d=2$, ω_z i.i.d., >2 moments

ω_z has a continuous CDF

Passage Time:

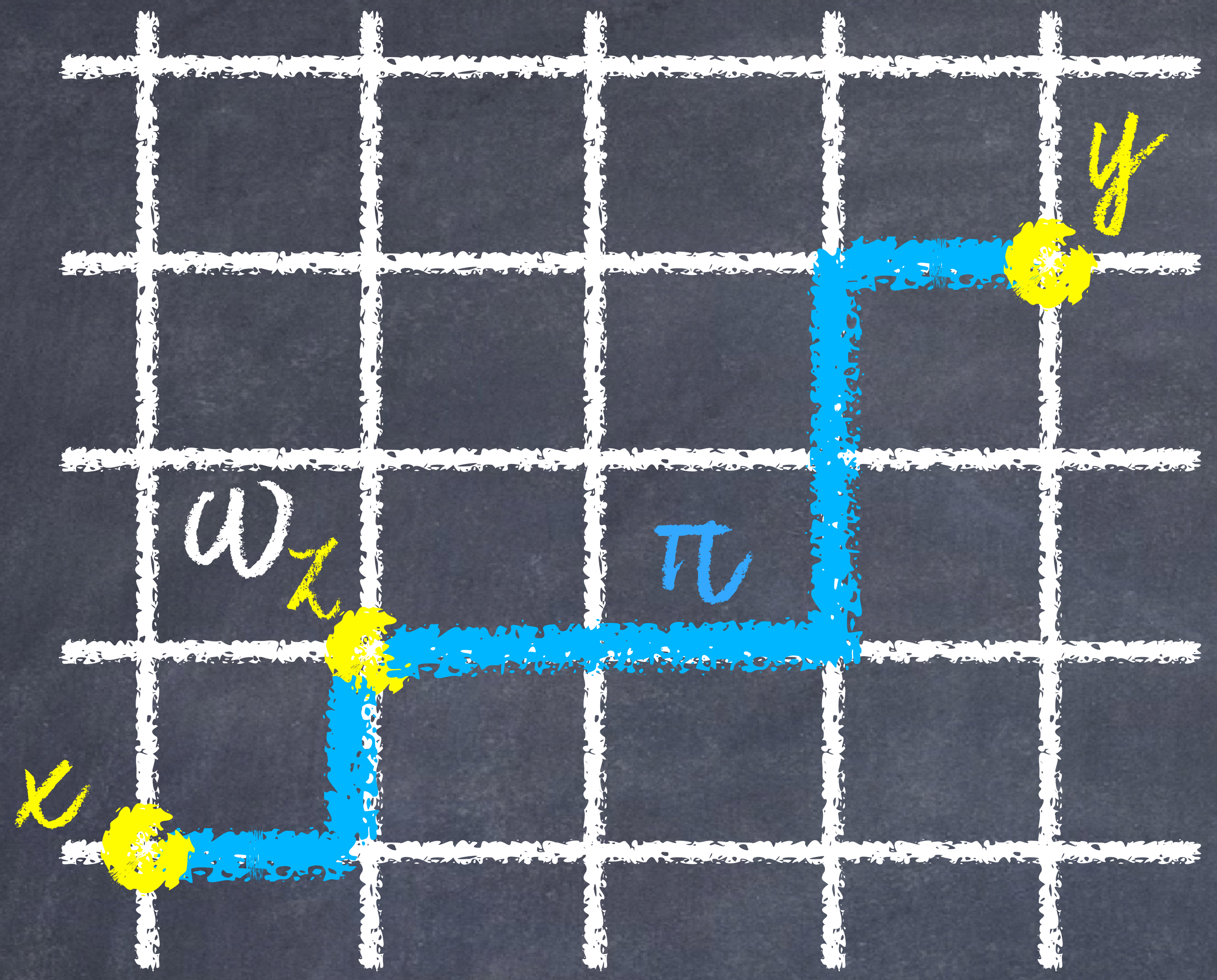


$d=2$, ω_z i.i.d., >2 moments

ω_z has a continuous CDF

Passage Time:

$$\sum_{z\in\pi} \omega_z$$

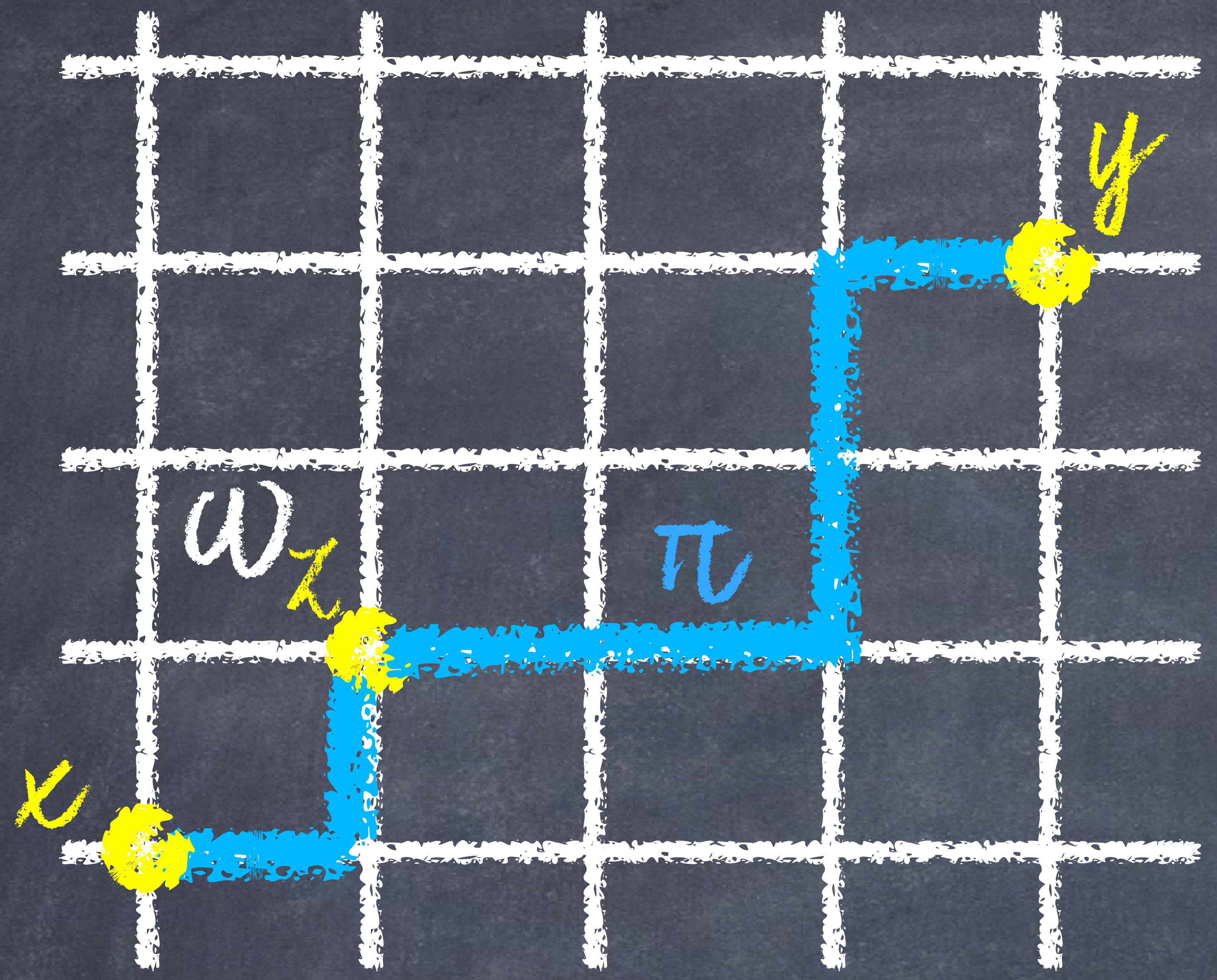


$d=2$, ω_z i.i.d., >2 moments

ω_z has a continuous CDF

Last Passage Time:

$$G_{xy} = \max_{\pi} \sum_{z \in \pi} \omega_z$$

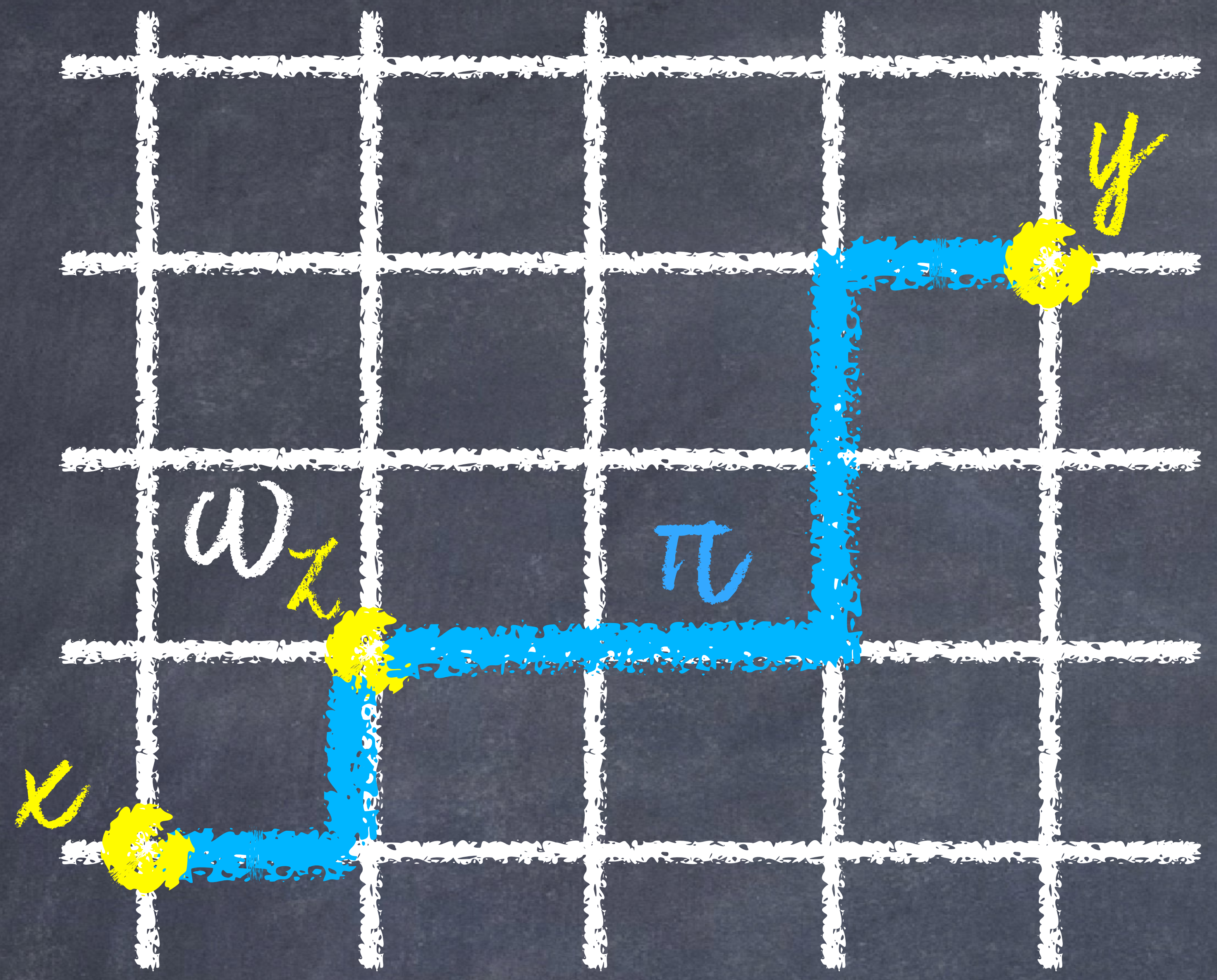


$d=2$, ω_z i.i.d., >2 moments

ω_z has a continuous CDF

Last Passage Time:

Point-to-point: $G_{xy} = \max_{\pi} \sum_{z \in \pi} \omega_z$



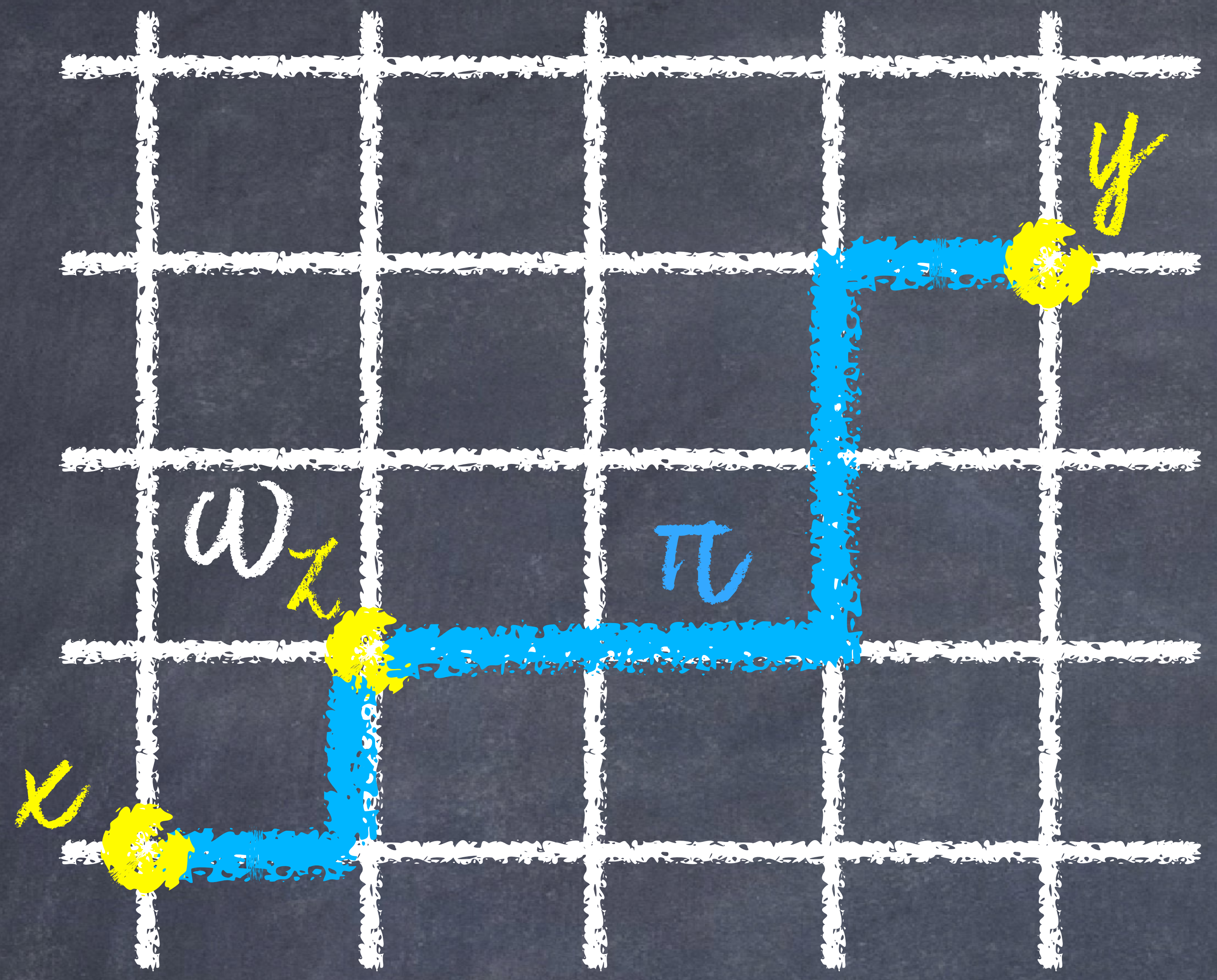
$d=2$, ω_z i.i.d., >2 moments

ω_z has a continuous CDF

Last Passage Time:

Point-to-point: $G_{xy} = \max_{\pi} \sum_{z \in \pi} \omega_z$

Geodesic: Maximizing path (unique)



$d=2$, ω_z i.i.d., >2 moments

ω_z has a continuous CDF

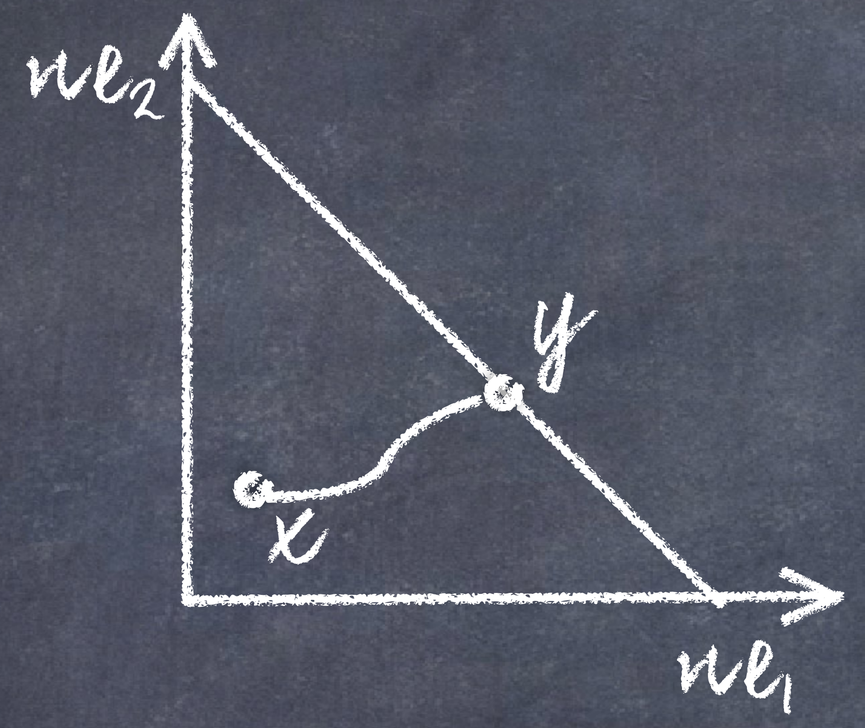
Last Passage Time:

Point-to-point: $G_{xy} = \max_{\pi} \sum_{z \in \pi} \omega_z$

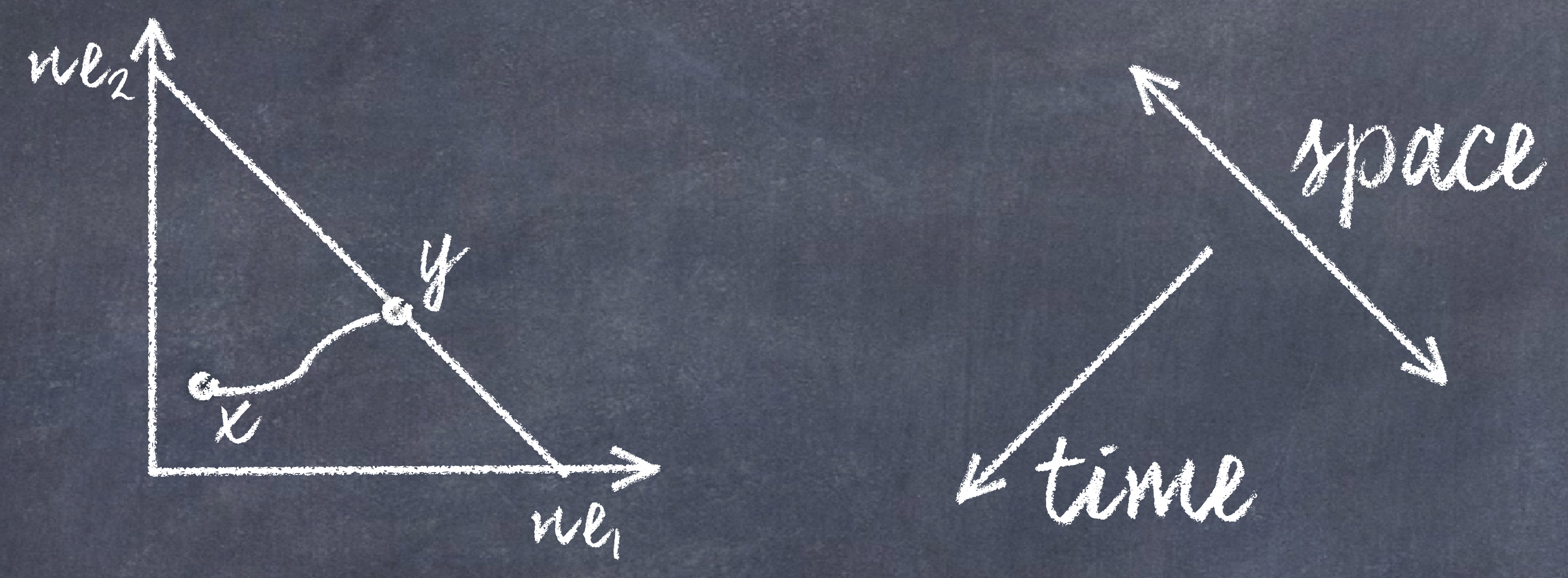
Geodesic: Maximizing path (unique)

Semi-infinite geodesic: each finite piece is a geodesic

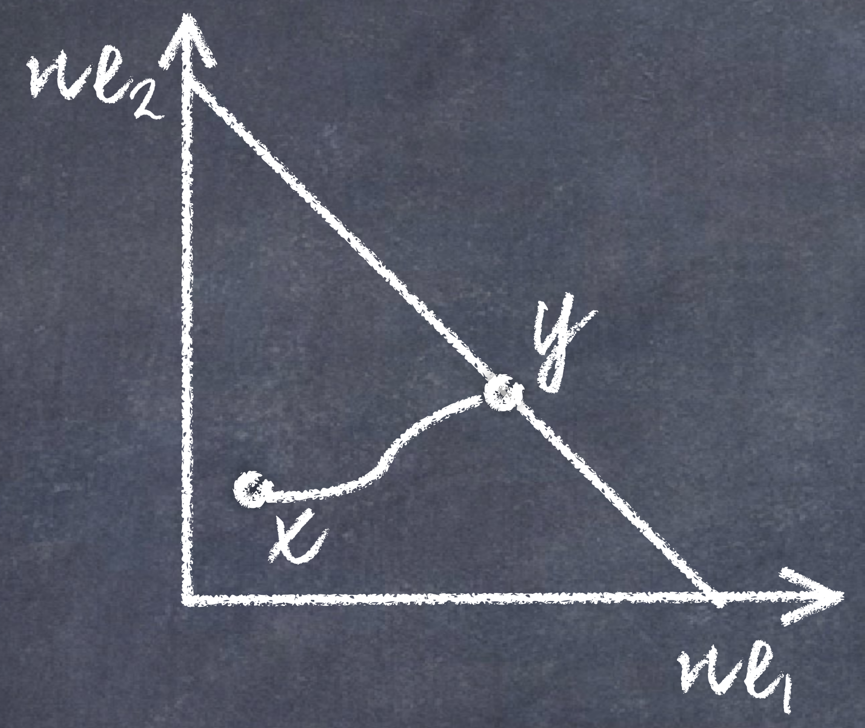
Point-to-line: $G_{x,(n)}(h) = \max_{\pi,y} \{h\cdot(y-x)+\sum \omega_z\}$

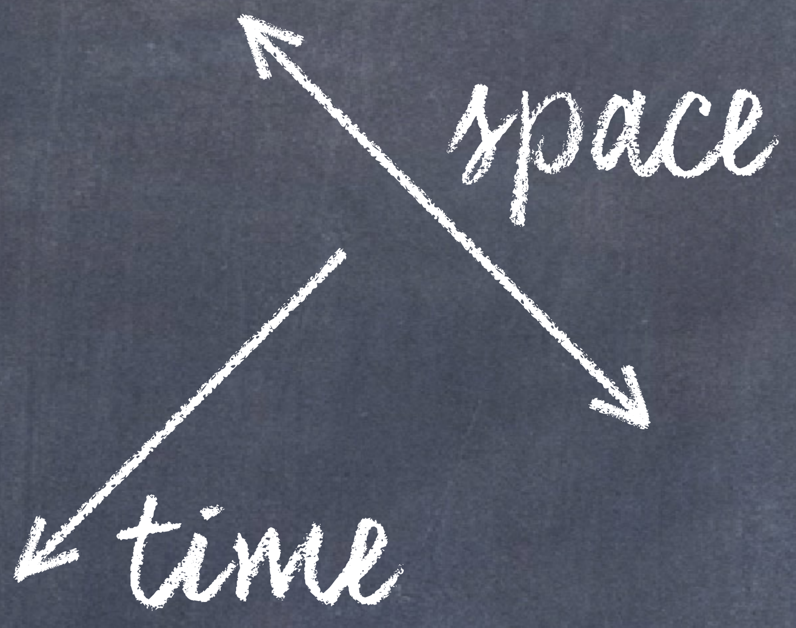


Point-to-line: $G_{x,(n)}(h) = \max_{\pi,y} \{h\cdot(y-x)+\sum \omega_z\}$

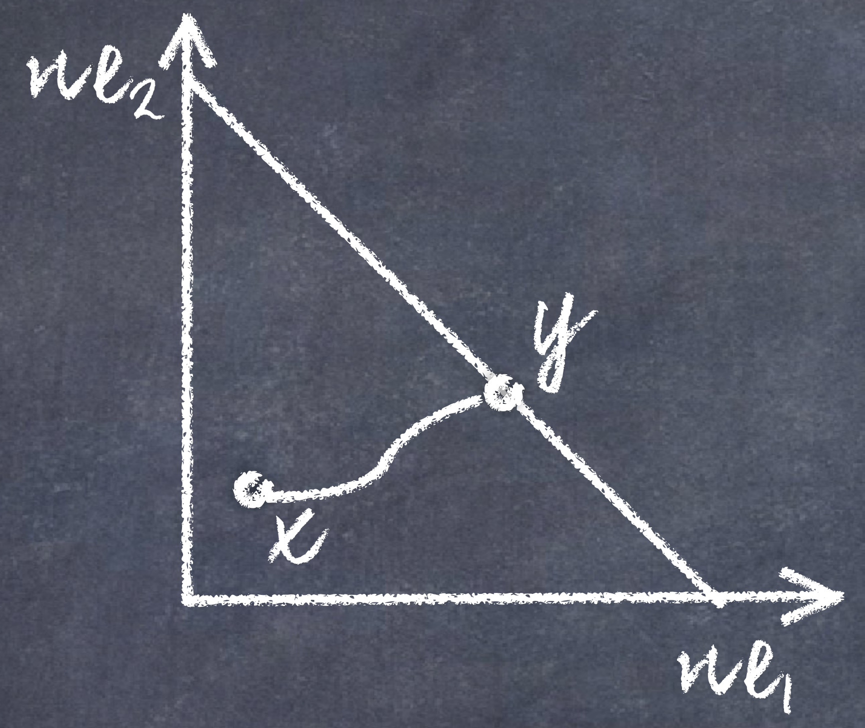


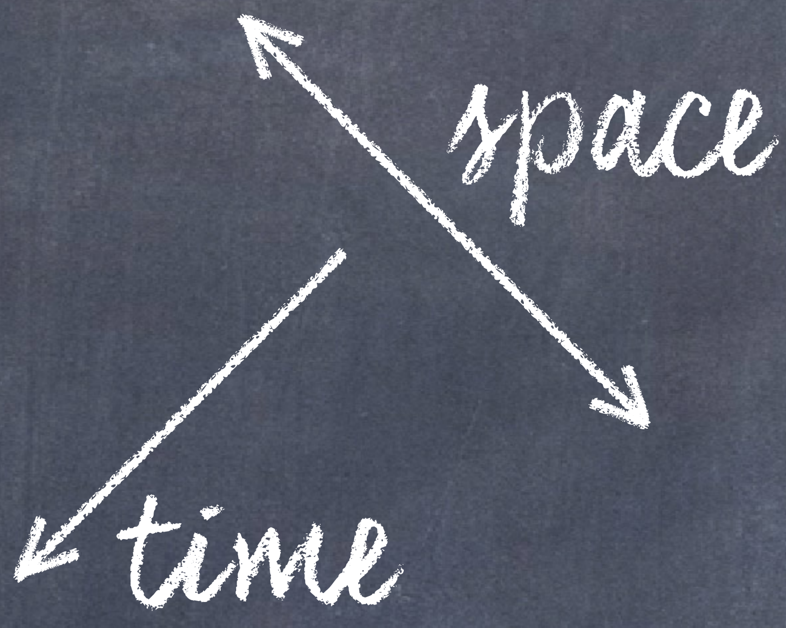
Point-to-line: $G_{x,(n)}(h) = \max_{\pi,y} \{ \overbrace{h\cdot(y-x)}^{\text{linear initial condition}} + \underbrace{\sum \omega_z}_{\text{Action}} \}$





Point-to-line: $$G_{x,(n)}(h) \overset{\text{Lax-Oleinik}}{=} \max_{\pi,y} \Big\{ \overbrace{h\cdot(y-x)}^{\text{linear initial condition}} + \underbrace{\sum \omega_z}_{\text{Action}} \Big\}$$

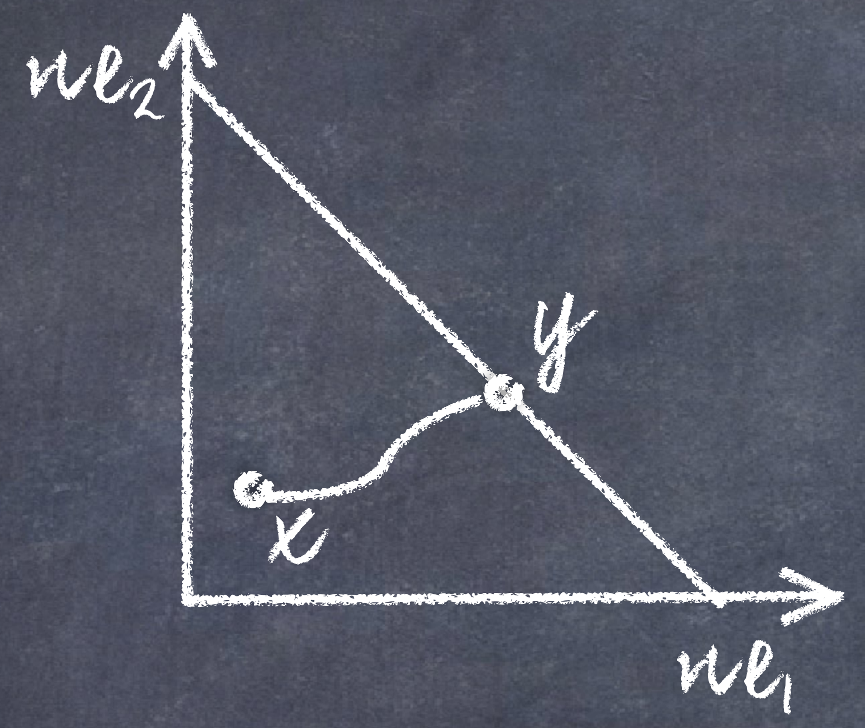


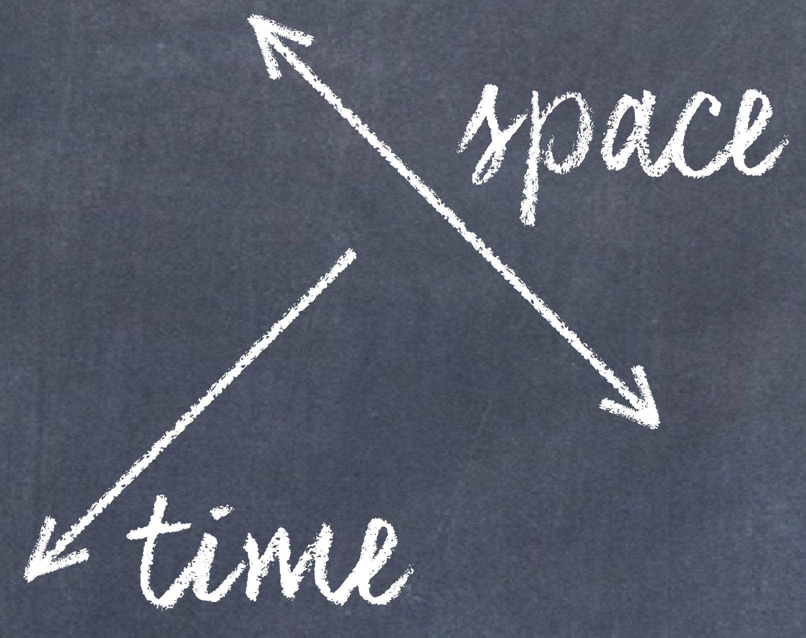


Lax-Oleinik

Point-to-line: $G_{x,(n)}(h) = \max_{\pi,y} \Big\{ h\cdot \overbrace{(y-x)}^{\text{linear initial condition}} + \underbrace{\sum \omega_z}_{\text{Action}} \Big\}$

conserved quantity

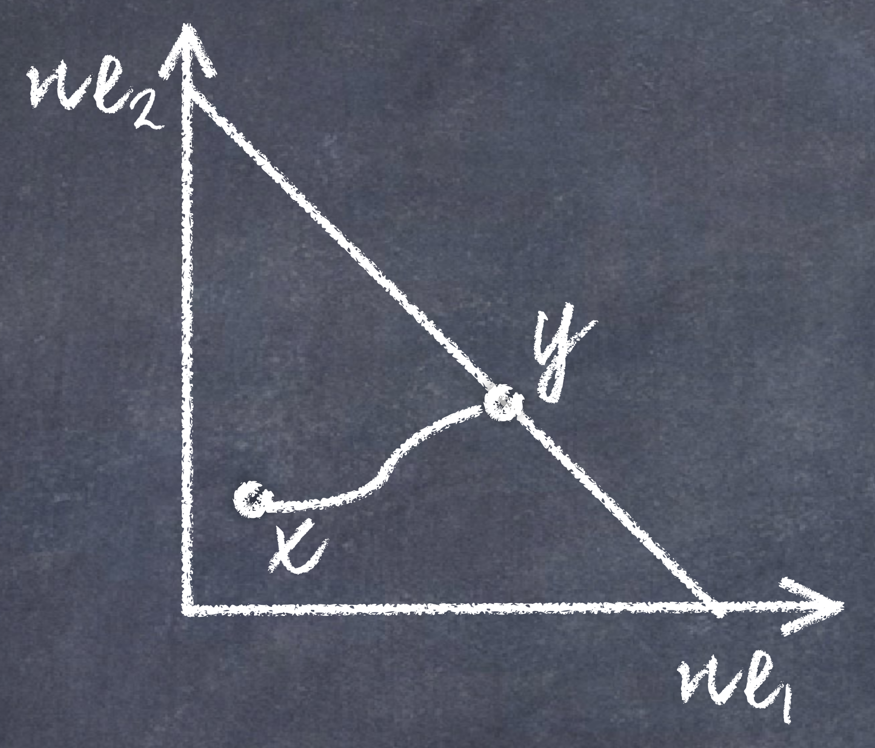


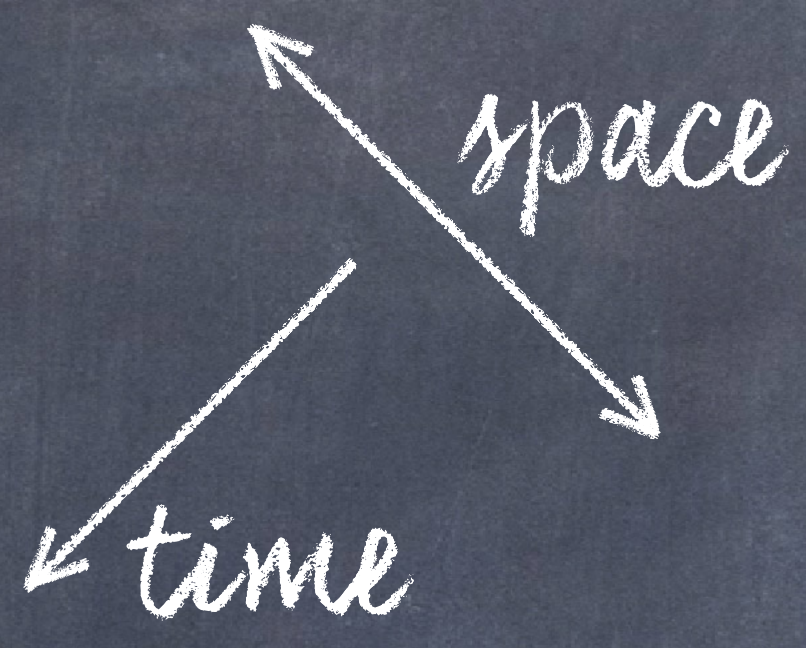


Lax-Oleinik

Point-to-line: $G_{x,(n)}(h) = \max_{\pi,y}\left\{\overbrace{h\cdot(y-x)}^{\text{linear initial condition}}+\underbrace{\sum\omega_z}_{\text{Action}}\right\}$

conserved quantity





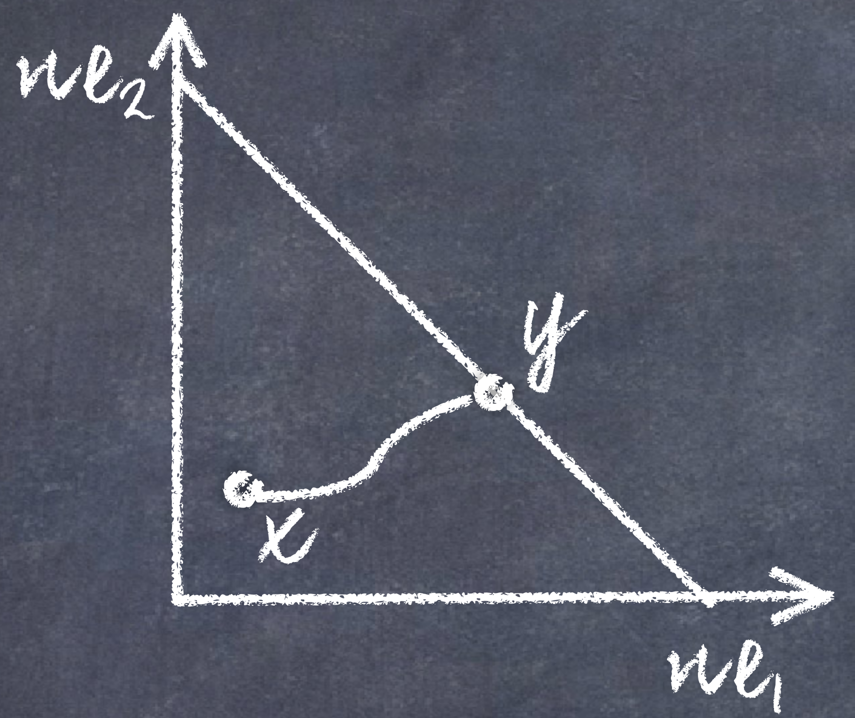
Both $G(x)=G_{xy}$ and $G_{x,(n)}(h)$ satisfy

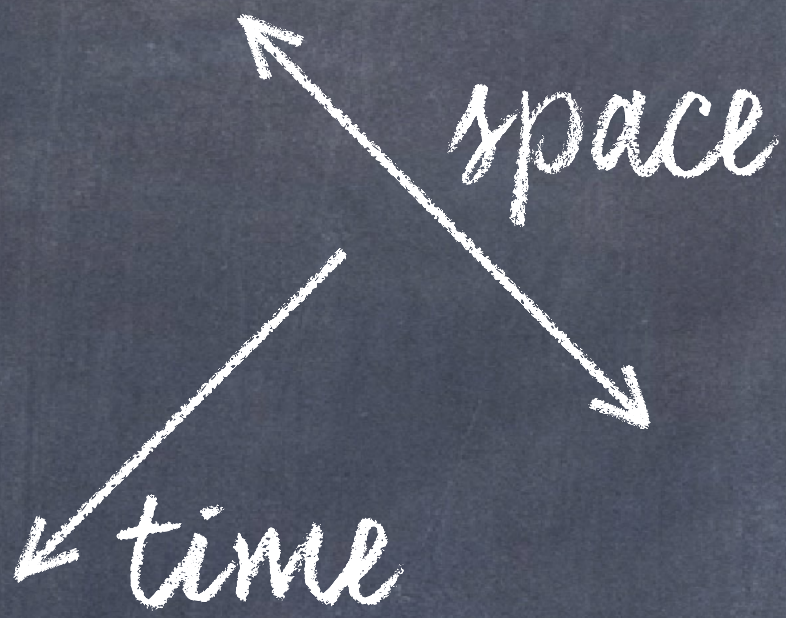
$$G(x)=\omega_x+G(x+e_1)\vee G(x+e_2)$$

Lax-Oleinik

Point-to-line: $G_{x,(n)}(h) = \max_{\pi,y} \{ h \cdot \overbrace{(y-x)}^{\text{linear initial condition}} + \underbrace{\sum \omega_z}_{\text{Action}} \}$

conserved quantity





Both $G(x) = G_{xy}$ and $G_{x,(n)}(h)$ satisfy

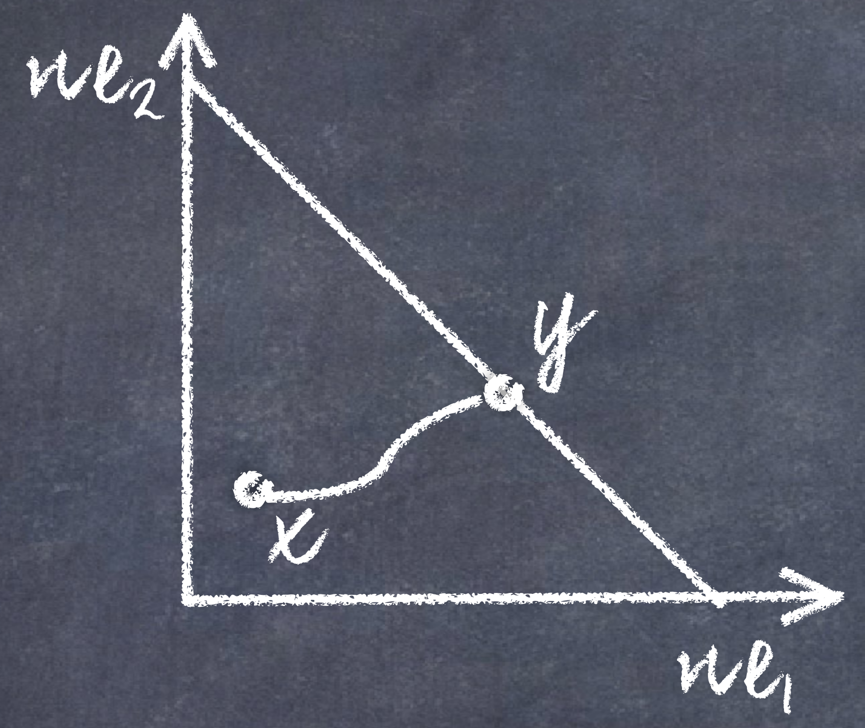
$$G(x) = \omega_x + G(x+e_1) \vee G(x+e_2)$$

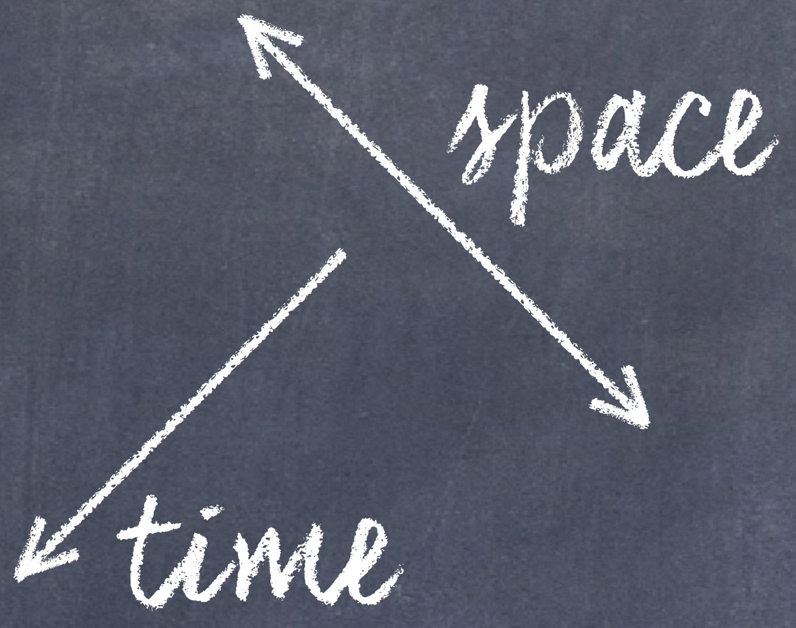
Probabilistic discretization of HJB

Lax-Oleinik

Point-to-line: $$G_{x,(n)}(h) = \max_{\pi,y}\Big\{\overbrace{h\cdot(y-x)}^{\text{linear initial condition}} + \underbrace{\sum_{z\in\pi}\omega_z}_{\text{Action}}\Big\}$$

conserved quantity





Both $G(x)=G_{xy}$ and $G_{x,(n)}(h)$ satisfy

$$G(x) = \omega_x + G(x+e_1) \vee G(x+e_2)$$

Probabilistic discretization of HJB

Shape Theorem:

Shape Theorem:

$\exists\, g : \mathbb{R}^2_+ \to \mathbb{R}$ deterministic, concave, homogenous

Shape Theorem:

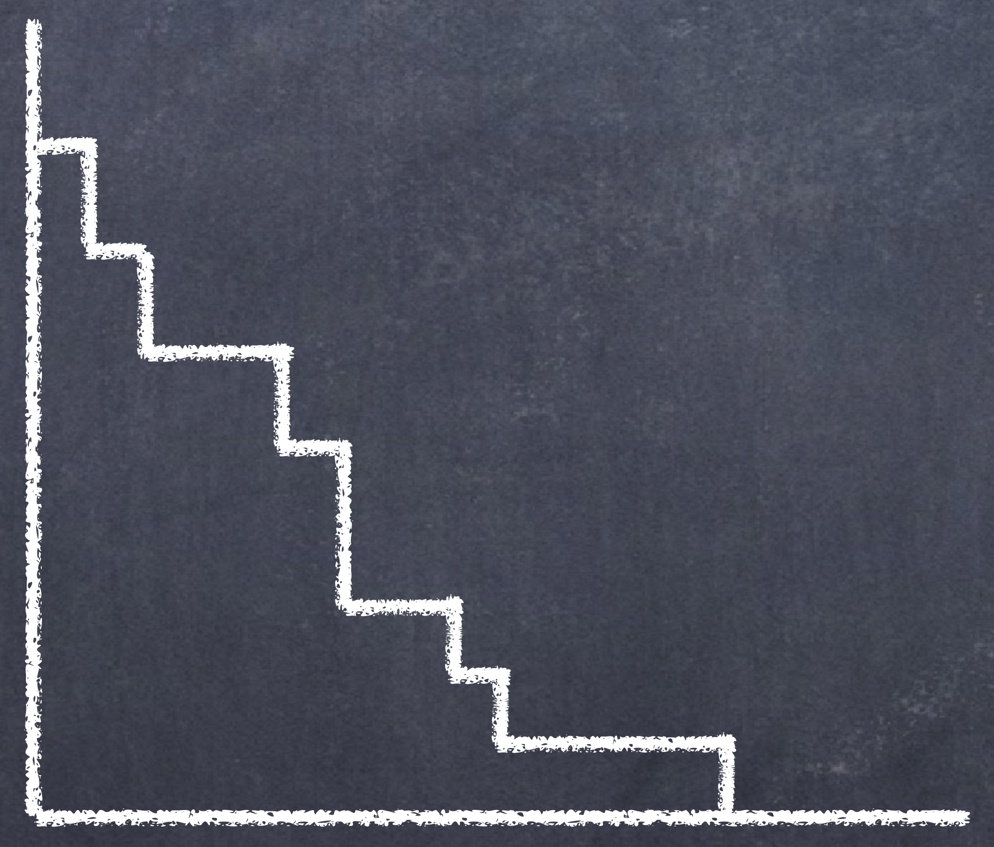
$\exists\, g : \mathbb{R}_+^2 \to \mathbb{R}$ deterministic, concave, homogenous

$$\limsup_{|x|\to\infty} \frac{|G_{0x} - g(x)|}{|x|} = 0 \quad \text{almost surely}$$

Shape Theorem:

$\exists\, g: \mathbb{R}_+^2 \to \mathbb{R}$ deterministic, concave, homogenous

$$\limsup_{|x|\to\infty} \frac{|G_{0x} - g(x)|}{|x|} = 0 \quad \text{almost surely}$$

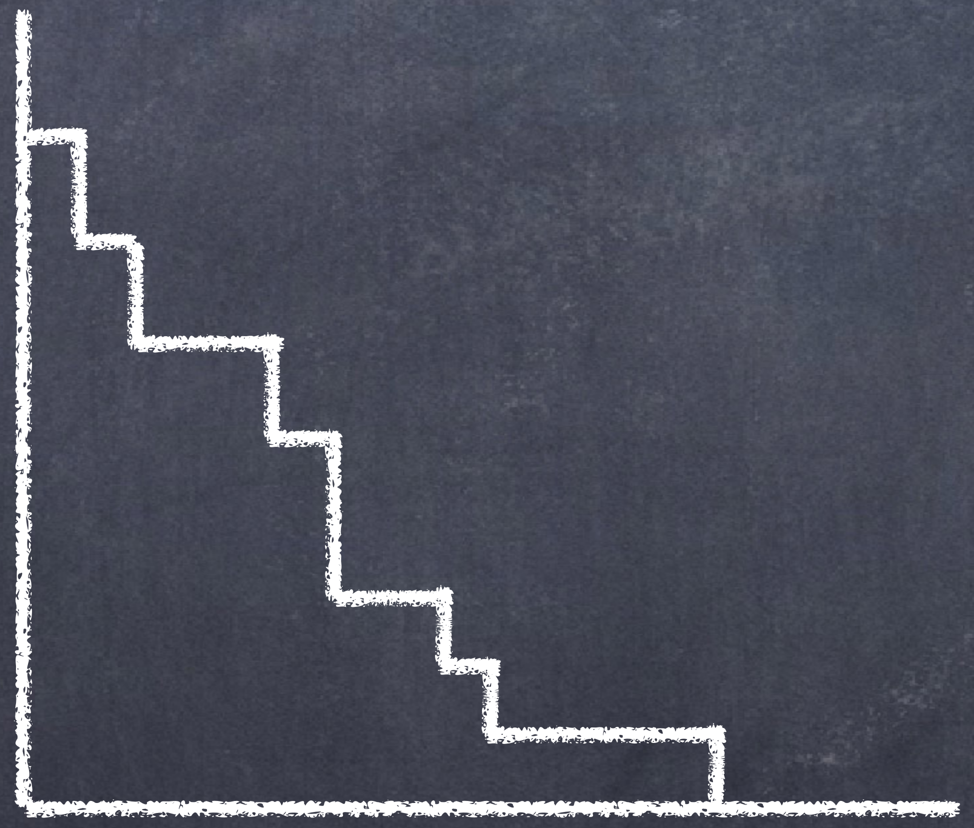


$\{x : G_{0x} \le t\}$

Shape Theorem:

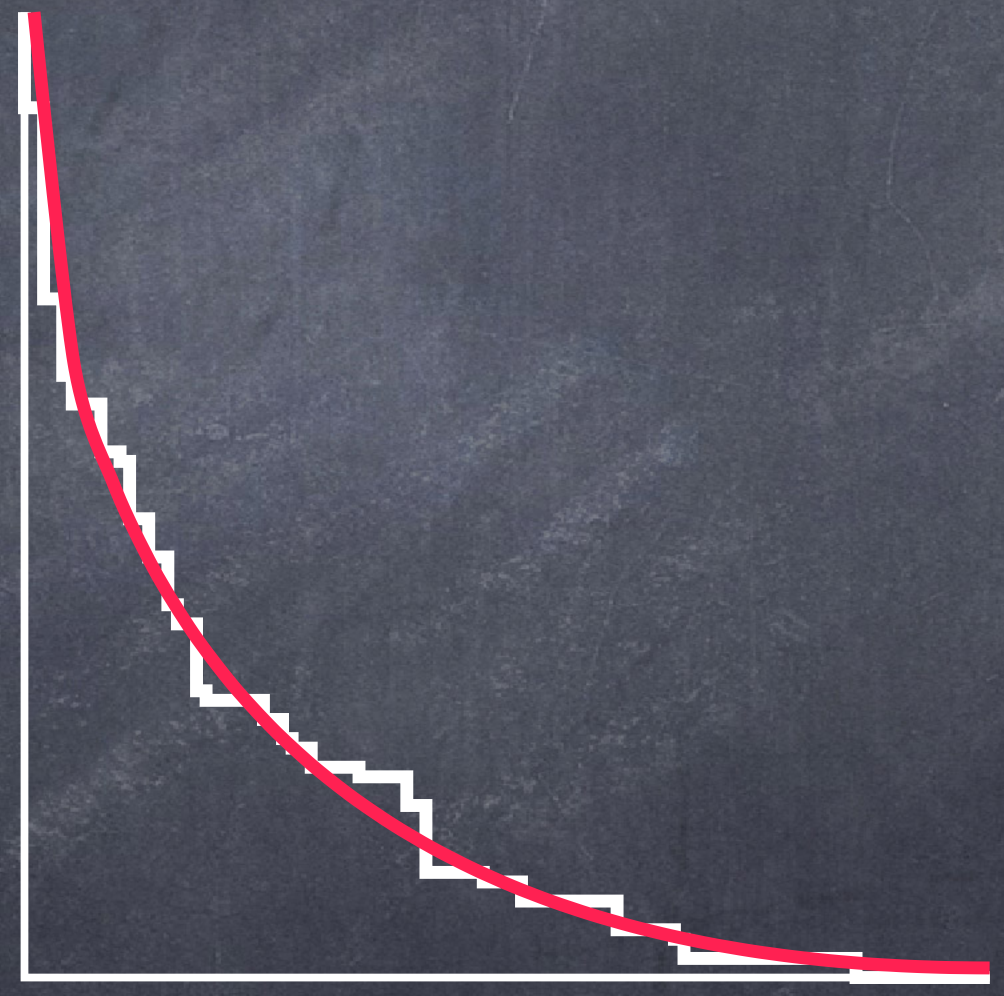
$\exists\, g : \mathbb{R}_+^2 \to \mathbb{R}$ deterministic, concave, homogenous

$$\limsup_{|x|\to\infty} \frac{|G_{0x} - g(x)|}{|x|} = 0 \quad \text{almost surely}$$



$\{x : G_{0x} \le t\}$

$$\Big/ \, t \quad \xrightarrow[t\to\infty]{}$$



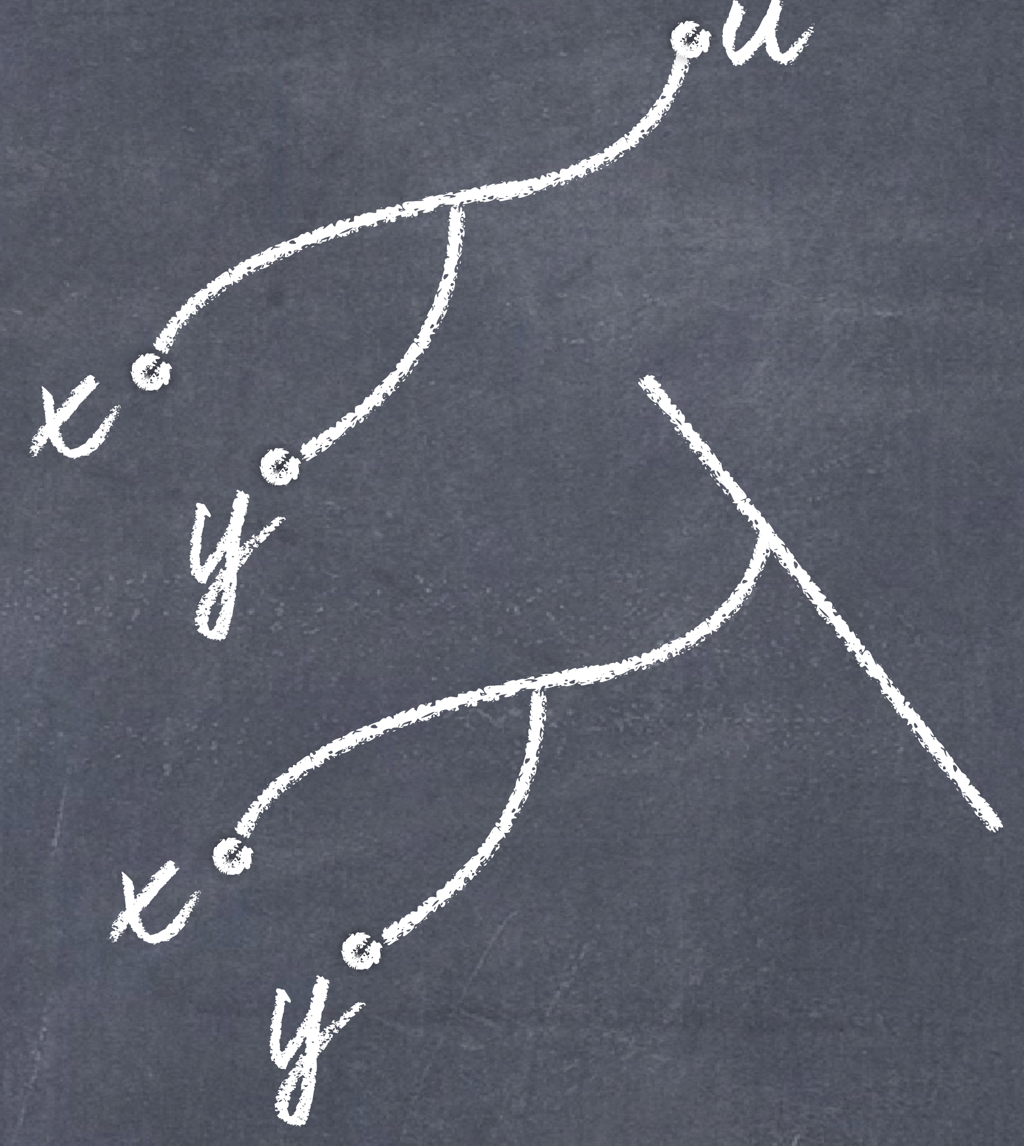
$\{\xi : g(\xi) \le 1\}$

Busemann functions:

Busemann functions: $B_u(x,y) = G_{xu} - G_{yu}$

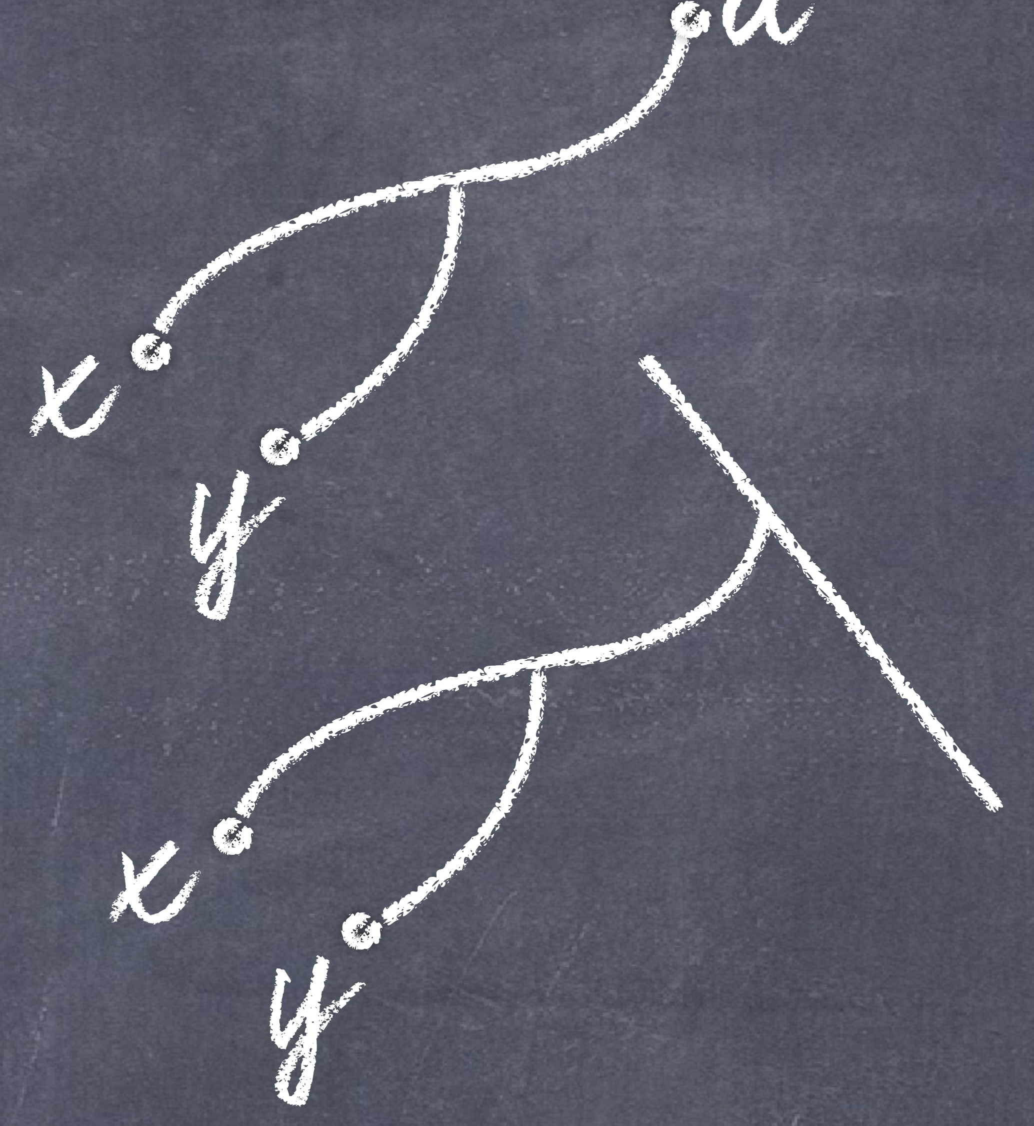
Busemann functions: $B_u(x,y) = G_{xu} - G_{yu}$

or $B_n^h(x,y) = G_{x(n)}(h) - G_{y(n)}(h)$



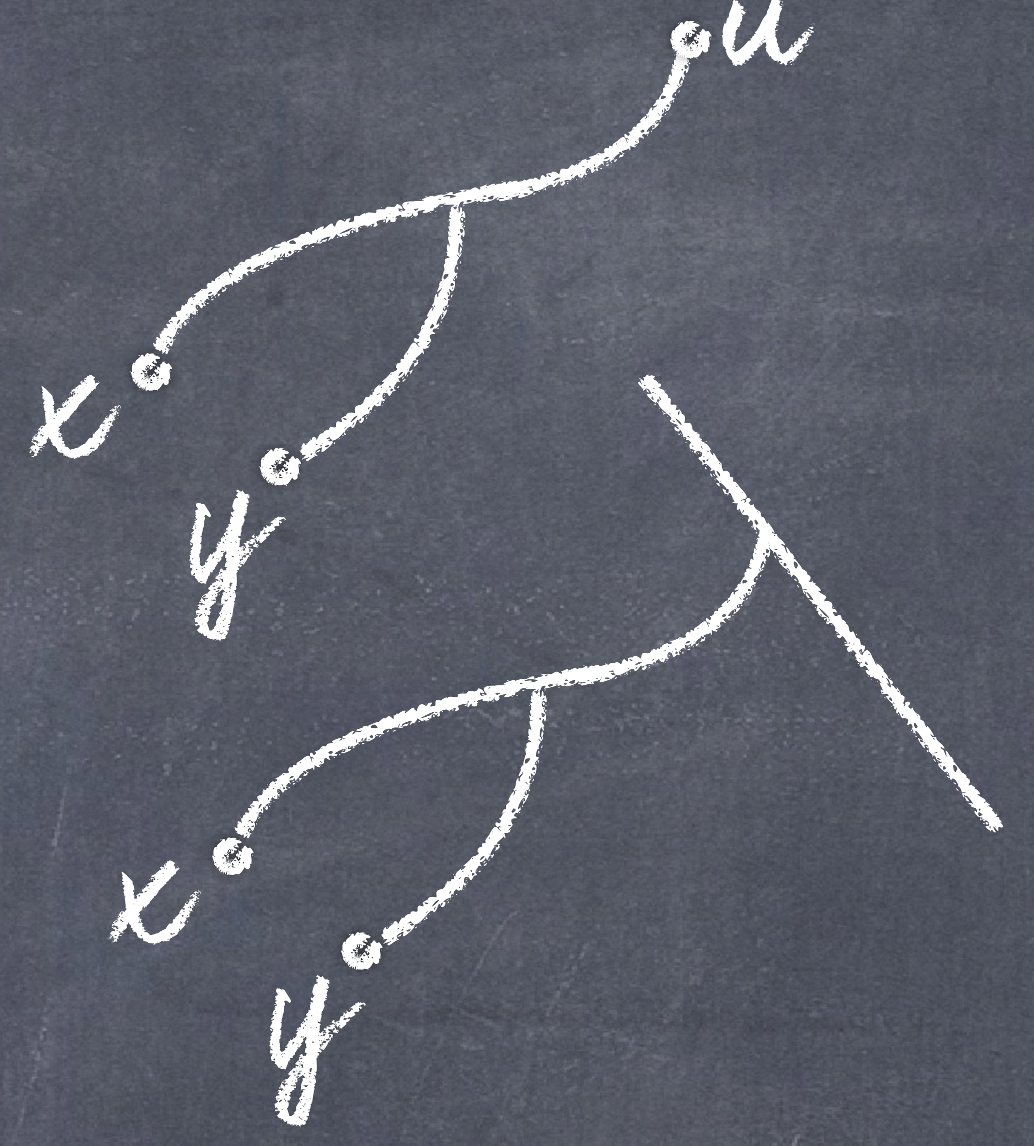
Busemann functions: $B_u(x,y) = G_{xu} - G_{yu}$

Solution to discrete Burgers $\longleftrightarrow$ or $B_n^h(x,y) = G_{x(n)}(h) - G_{y(n)}(h)$



Busemann functions: $B_u(x,y) = G_{xu} - G_{yu}$

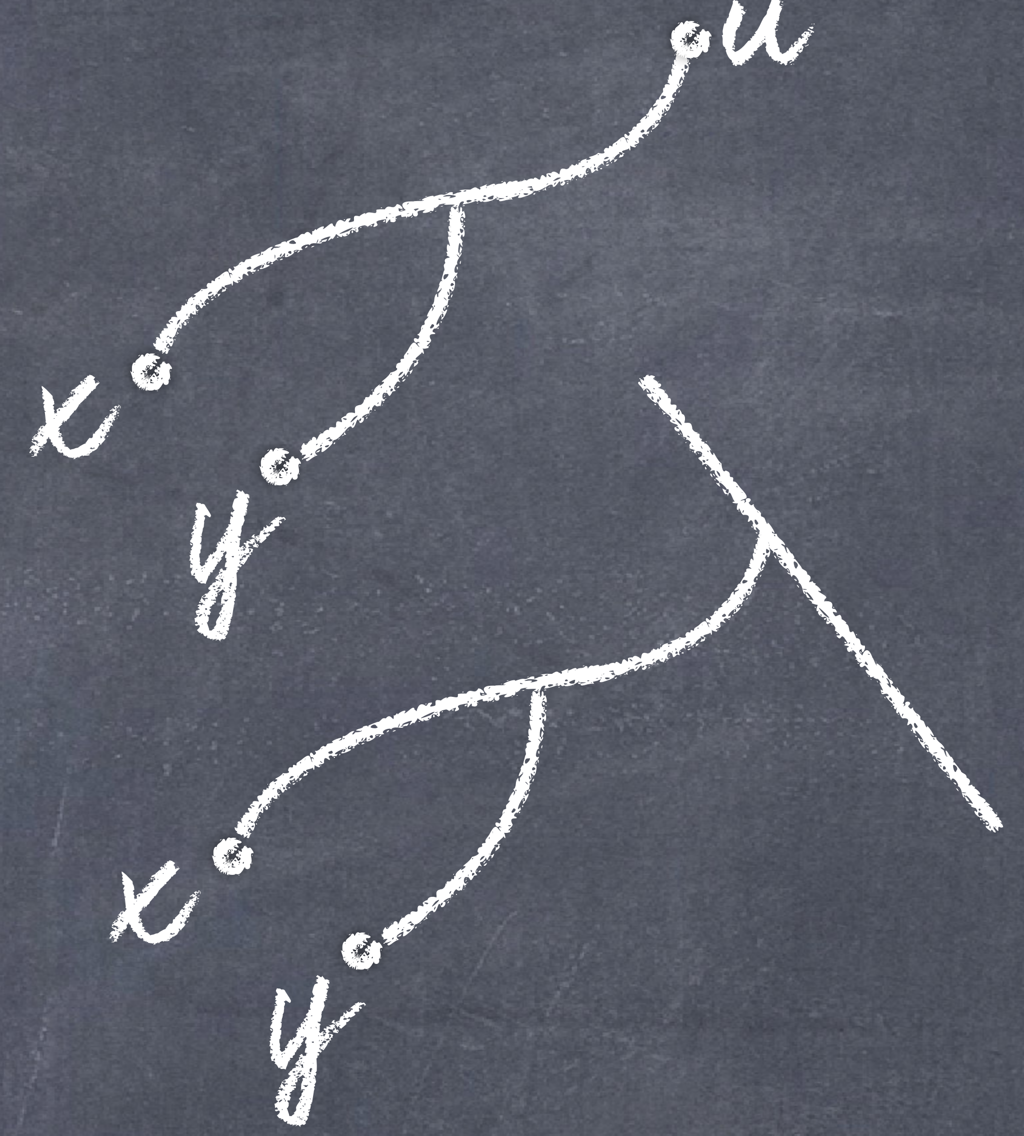
Solution to discrete Burgers $\longleftrightarrow$ or $B_n^h(x,y) = G_{x(n)}(h) - G_{y(n)}(h)$



Cocycle: $B(x,y) + B(y,z) = B(x,z)$

Busemann functions: $B_u(x,y)=G_{xu}-G_{yu}$

Solution to discrete Burgers $\longleftrightarrow$ or $B^h_n(x,y)=G_{x(n)}(h)-G_{y(n)}(h)$



Cocycle: $B(x,y)+B(y,z)=B(x,z)$

Recovery: $B(x,x+e_1)\wedge B(x,x+e_2)=\omega_x$

Busemann functions: $B_u(x,y) = G_{xu} - G_{yu}$

Solution to discrete Burgers $\longleftrightarrow$ or $B_n^h(x,y) = G_{x(n)}(h) - G_{y(n)}(h)$

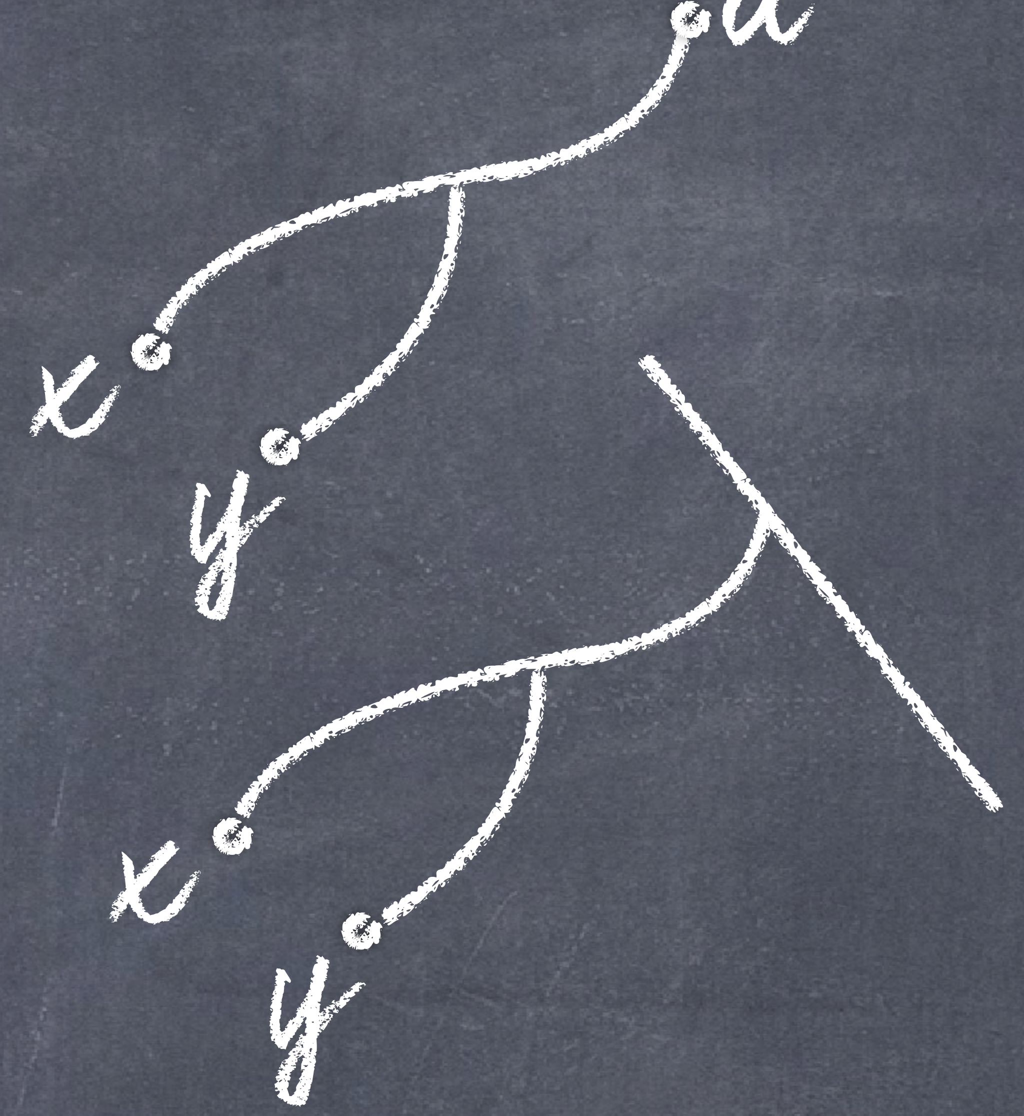
Cocycle: $B(x,y)+B(y,z)=B(x,z)$

Recovery: $B(x,x+e_1)\wedge B(x,x+e_2)=\omega_x$

$G(x)=\omega_x+G(x+e_1)\vee G(x+e_2)$

__Busemann functions:__ $B_u(x,y) = G_{xu} - G_{yu}$

Solution to discrete Burgers $\longleftrightarrow$ or $B_n^h(x,y) = G_{x(n)}(h) - G_{y(n)}(h)$



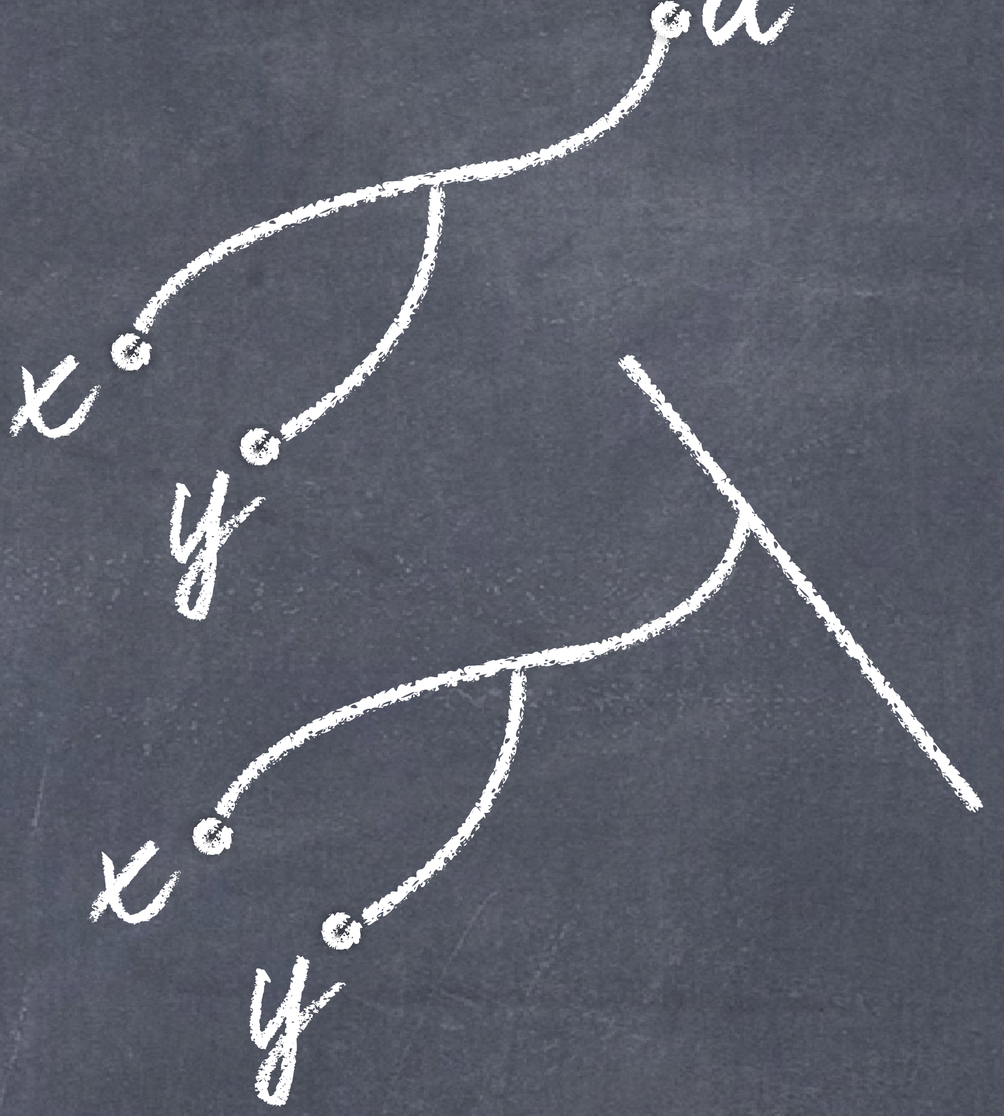
__Cocycle:__ $B(x,y) + B(y,z) = B(x,z)$

__Recovery:__ $B(x,x+e_1) \wedge B(x,x+e_2) = \omega_x$

$$G(x) = \omega_x + G(x+e_1) \vee G(x+e_2)$$

Busemann functions: $B_u(x,y) = G_{xu} - G_{yu}$

Solution to discrete Burgers $\longleftrightarrow$ or $B^h_n(x,y) = G_{x(n)}(h) - G_{y(n)}(h)$



Cocycle: $B(x,y) + B(y,z) = B(x,z)$

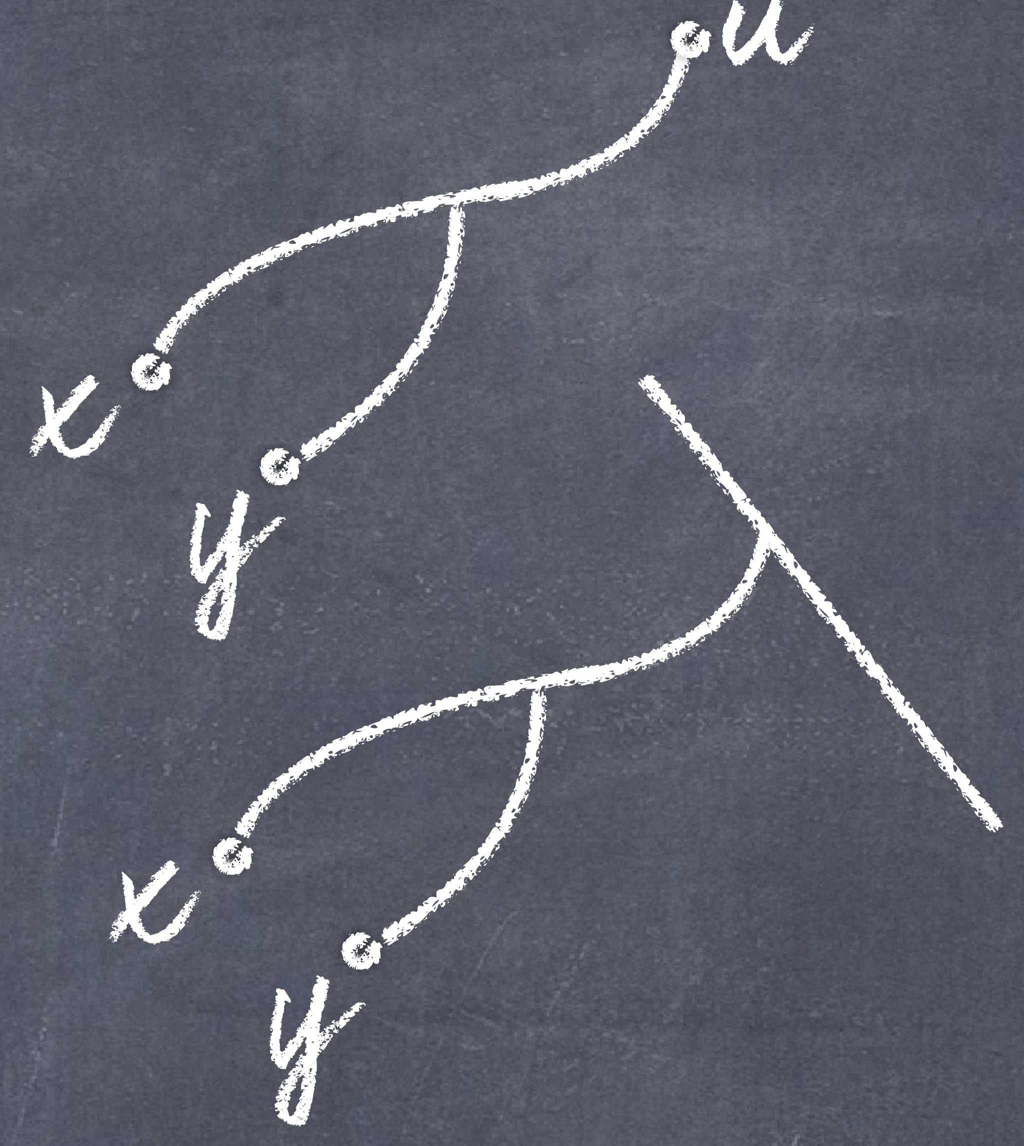
Recovery: $B(x,x+e_1) \wedge B(x,x+e_2) = \omega_x$

geodesic follows smallest increment

$$G(x) = \omega_x + G(x+e_1) \vee G(x+e_2)$$

Busemann functions: $B_u(x,y) = G_{xu} - G_{yu}$

Solution to discrete Burgers $\longleftrightarrow$ or $B^h_n(x,y) = G_{x(n)}(h) - G_{y(n)}(h)$



Cocycle: $B(x,y) + B(y,z) = B(x,z)$

Recovery: $B(x,x+e_1) \wedge B(x,x+e_2) = \omega_x$

geodesic follows smallest increment

$G(x) = \omega_x + G(x+e_1) \vee G(x+e_2)$

$\cdot v$

$\cdot u$

Monotonicity:

x° $\quad \overset{\circ}{x}+e_1$

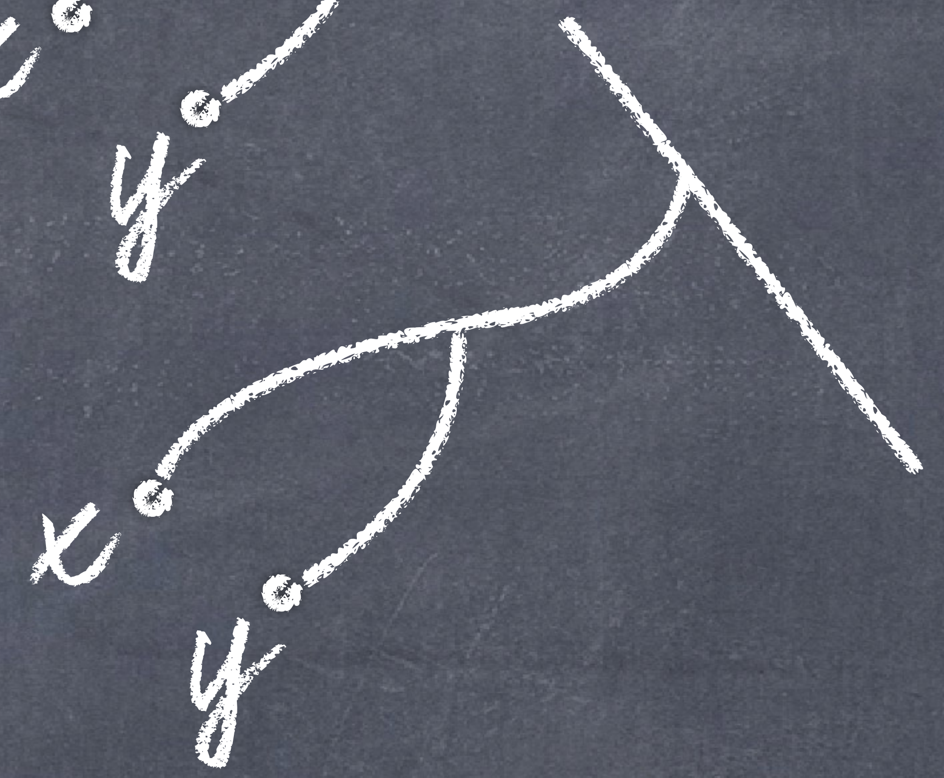
Busemann functions: $B_u(x,y) = G_{xu} - G_{yu}$

Solution to discrete Burgers $\longleftrightarrow$ or $B^h_n(x,y) = G_{x(n)}(h) - G_{y(n)}(h)$

Cocycle: $B(x,y)+B(y,z)=B(x,z)$

Recovery: $B(x,x+e_1)\wedge B(x,x+e_2)=\omega_x$

geodesic follows smallest increment

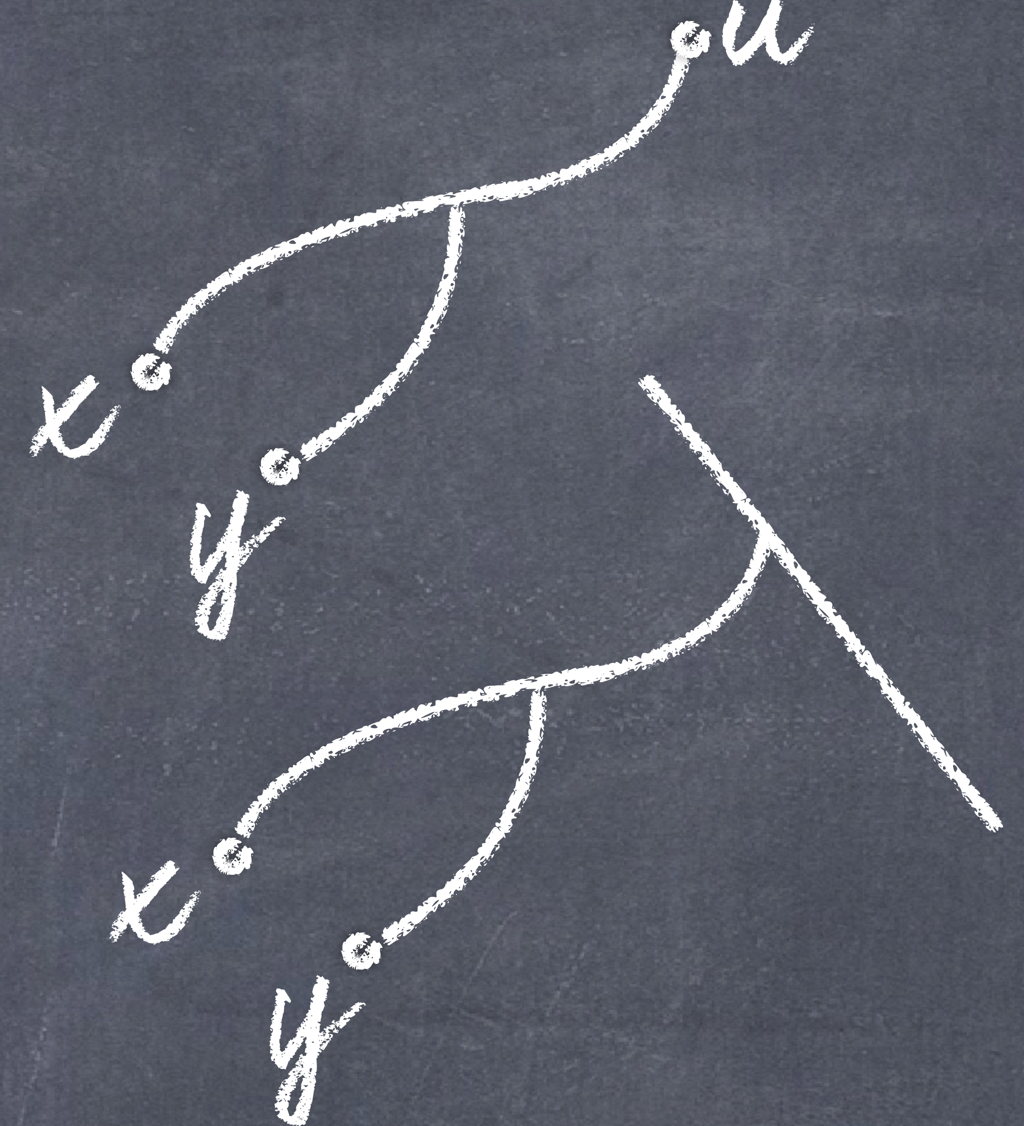


$$G(x)=\omega_x+G(x+e_1)\vee G(x+e_2)$$

Monotonicity:

Busemann functions: $B_u(x,y)=G_{xu}-G_{yu}$

Solution to discrete Burgers $\longleftrightarrow$ or $B^h_n(x,y)=G_{x(n)}(h)-G_{y(n)}(h)$



Cocycle: $B(x,y)+B(y,z)=B(x,z)$

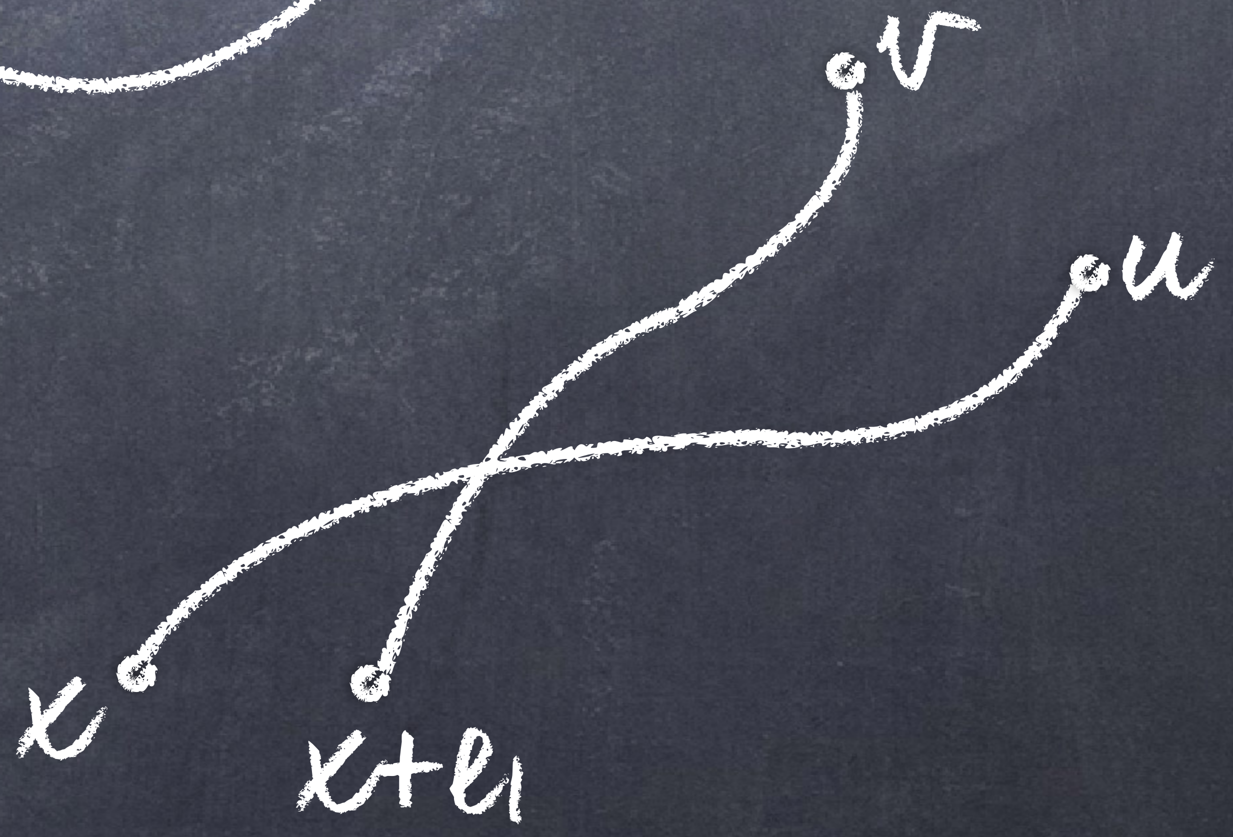
Recovery: $B(x,x+e_1)\wedge B(x,x+e_2)=\omega_x$

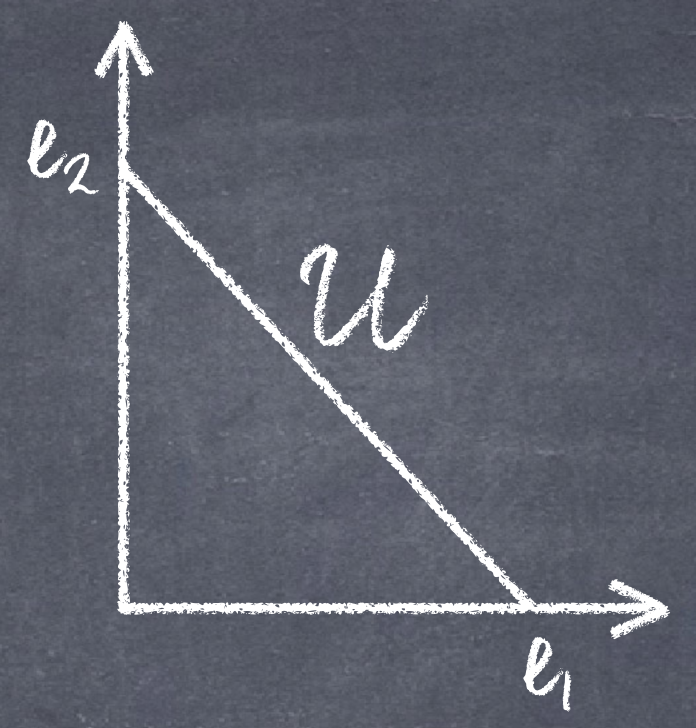
geodesic follows smallest increment

$$G(x)=\omega_x+G(x+e_1)\vee G(x+e_2)$$

Monotonicity: $B_u(x,x+e_1)\le B_v(x,x+e_1)$

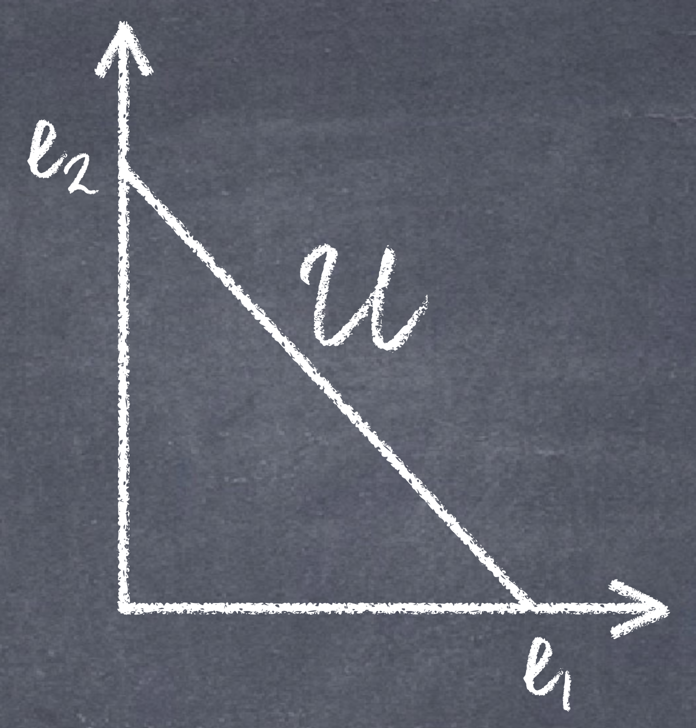
$B_u(x,x+e_2)\ge B_v(x,x+e_2)$



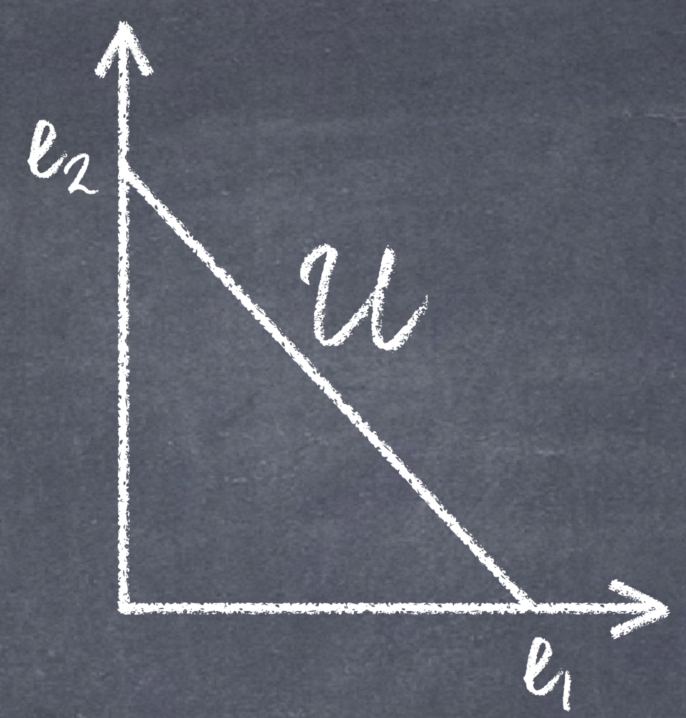


e_2
$\mathcal{U}$
e_1

Th. Fix "nice" $\xi \in \mathcal{U}$. Then a.s. $B_{x_n}(x,y) \to B^{\xi}(x,y)$ as $\frac{x_n}{n} \to \xi$



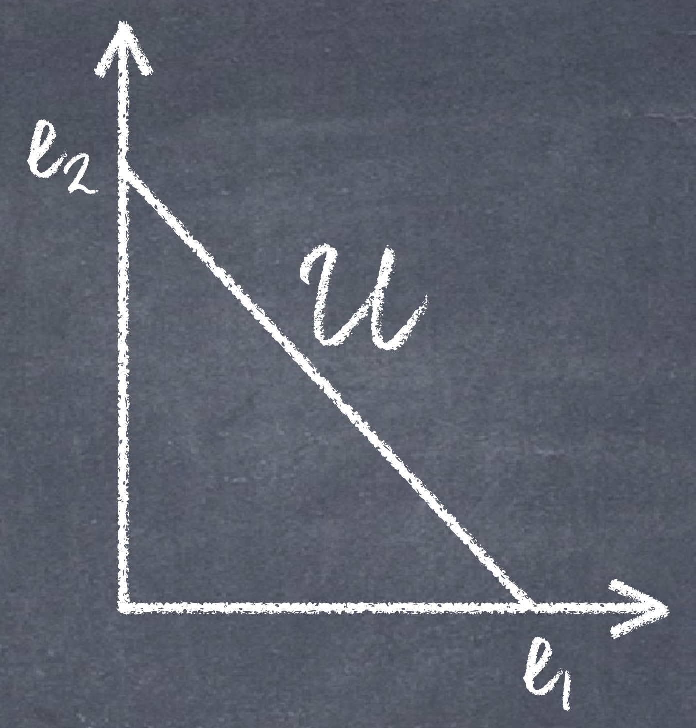
Th. Fix "nice" $\xi \in \mathcal{U}$. Then a.s. $B_{x_n}(x,y) \to B^{\xi}(x,y)$ as $\frac{x_n}{n} \to \xi$



$h = -\nabla g(\xi)$ (dual)

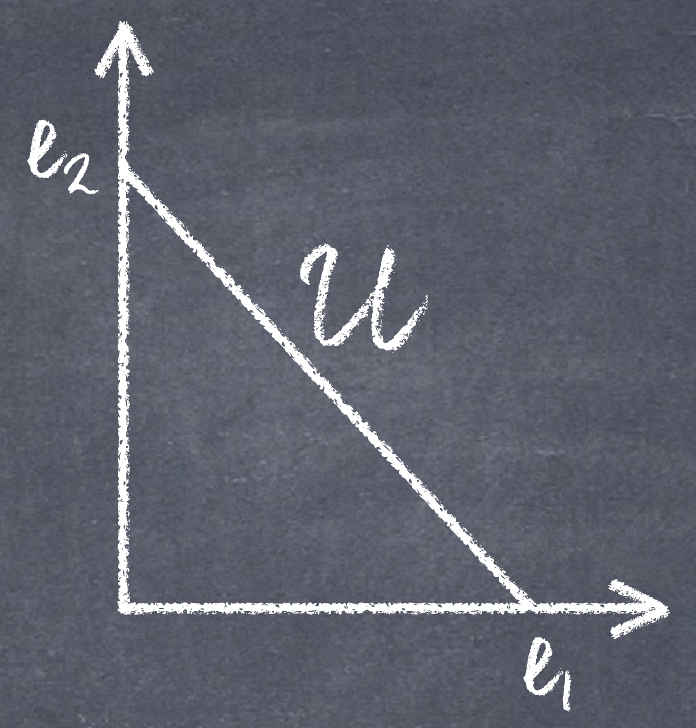
Th. Fix "nice" $\xi \in \mathcal{U}$. Then a.s. $B_{x_n}(x,y) \to B^{\xi}(x,y)$ as $\frac{x_n}{n} \to \xi$

$h = -\nabla g(\xi)$ (dual) $B^h_n(x,y)$



Th. Fix "nice" $\xi \in \mathcal{U}$. Then a.s. $B_{x_n}(x,y) \to B^{\xi}(x,y)$ as $\frac{x_n}{n} \to \xi$

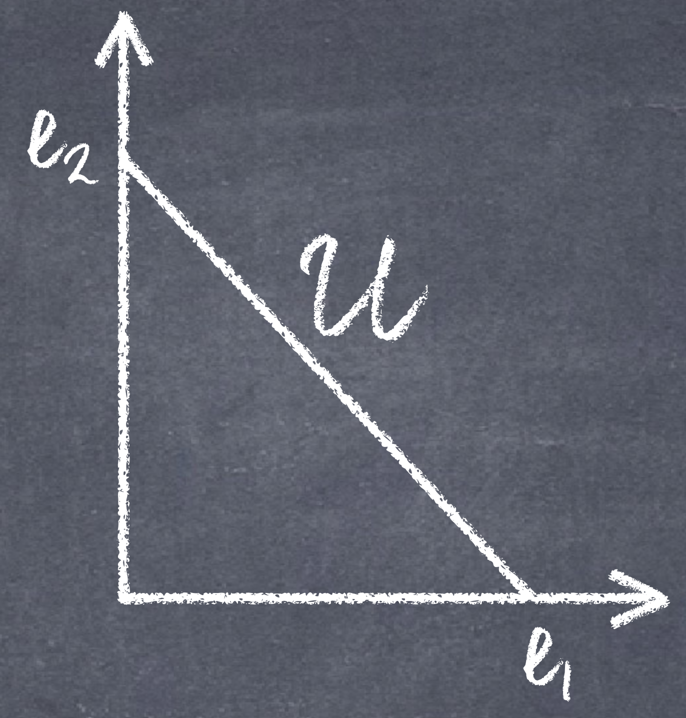
$h = -\nabla g(\xi)$ (dual) $\qquad B^h_n(x,y)$ $\qquad$ global solutions



Th. Fix "nice" $\xi \in \mathcal{U}$. Then a.s. $B_{x_n}(x,y) \to B^{\xi}(x,y)$ as $\frac{x_n}{n} \to \xi$

$h = -\nabla g(\xi)$ (dual) $\qquad B^h_n(x,y)$

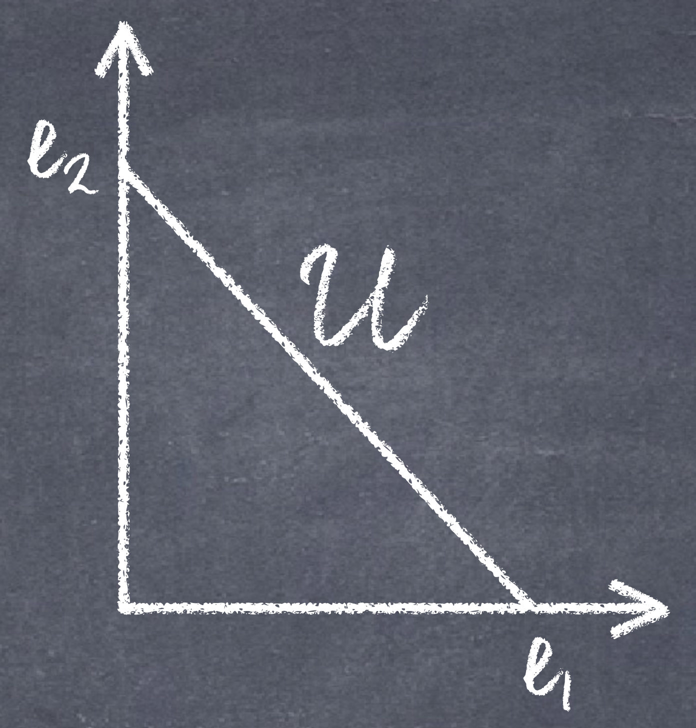
global solutions



Licea-Newman '96: for standard FPP

<u>Th.</u> Fix "nice" $\xi \in \mathcal{U}$. Then a.s. $B_{x_n}(x,y) \to B^{\xi}(x,y)$ as $\frac{x_n}{n} \to \xi$

$h = -\nabla g(\xi)$ (dual) $\qquad B^h_n(x,y)$ $\qquad$ global solutions



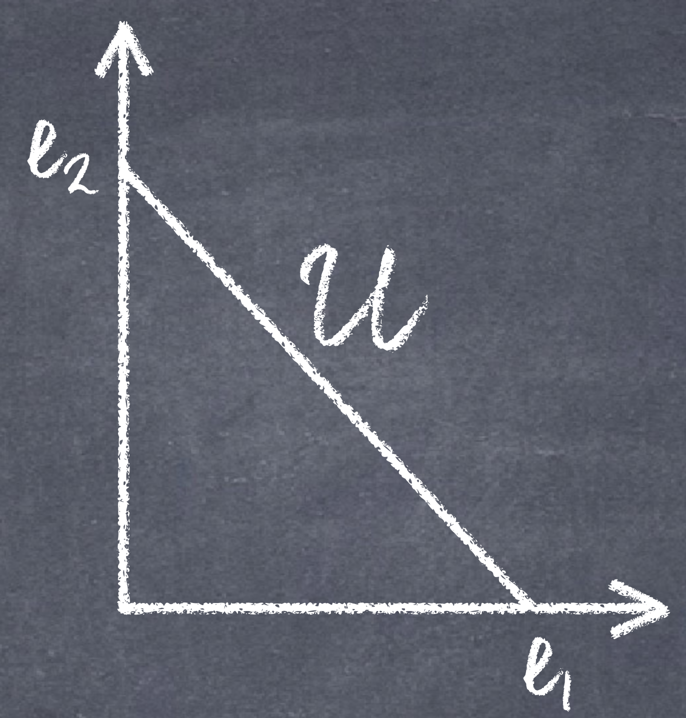
Licea-Newman '96: for standard FPP

curvature assumption on g

<u>Th.</u> Fix "nice" $\xi \in \mathcal{U}$. Then a.s. $B_{x_n}(x,y) \to B^{\xi}(x,y)$ as $\frac{x_n}{n} \to \xi$

$h = -\nabla g(\xi)$ (dual) $\quad B^h_n(x,y)$

global solutions

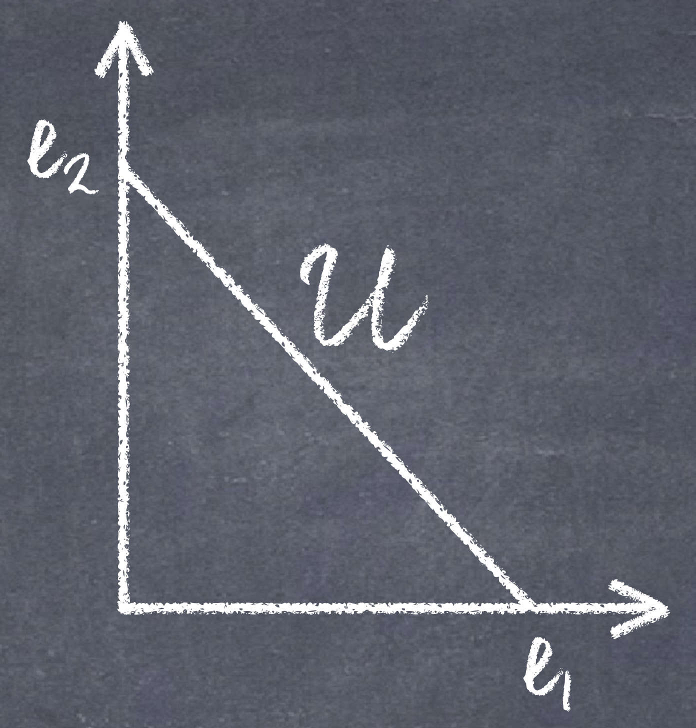


Licea-Newman '96: for standard FPP

curvature assumption on g

existence, uniqueness, coalescence of
ξ-directed semi-inf. geo. : $\frac{x_n}{|x_n|} \to \xi$

<u>Th.</u> Fix "nice" $\xi \in \mathcal{U}$. Then a.s. $B_{x_n}(x,y) \to B^{\xi}(x,y)$ as $\frac{x_n}{n} \to \xi$



$h = -\nabla g(\xi)$ (dual) $B^h_n(x,y)$

global solutions

Licea-Newman '96: for standard FPP

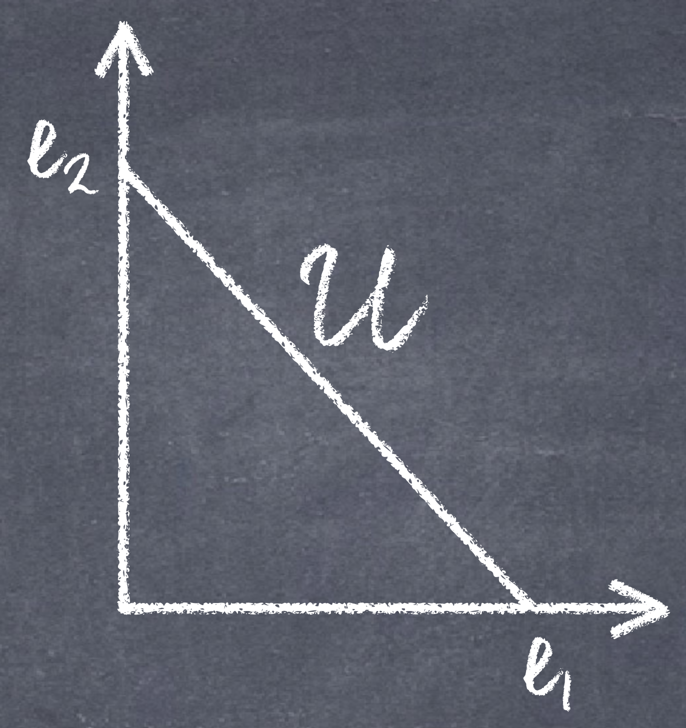
curvature assumption on g

existence, uniqueness, coalescence of
ξ-directed semi-inf. geo. : $\frac{x_n}{|x_n|} \to \xi$

Th. Fix "nice" $\xi \in \mathcal{U}$. Then a.s. $B_{x_n}(x,y) \to B^{\xi}(x,y)$ as $\frac{x_n}{n} \to \xi$

$h = -\nabla g(\xi)$ (dual) $\quad B^h_n(x,y) \nearrow$

global solutions



Licea-Newman '96: for standard FPP

curvature assumption on g

existence, uniqueness, coalescence of
ξ-directed semi-inf. geo. : $\frac{x_n}{|x_n|} \to \xi$

Coalescence implies $B_{x_n}(x,y) \to B^{\xi}(x,y)$

Conjecture: g differentiable, strictly concave, and satisfies the curvature assumption.

Conjecture: g differentiable, strictly concave, and satisfies the curvature assumption.

FPP: some non-lattice models with rotational symmetry
Euclidean FPP: Howard & Newman '97, '01

Conjecture: g differentiable, strictly concave, and satisfies the curvature assumption.

FPP: some non-lattice models with rotational symmetry
Euclidean FPP: Howard & Newman '97, '01

LPP: $\omega_x \sim \exp(1)$ gives $g(\xi) = (\sqrt{\xi_1} + \sqrt{\xi_2})^2$ (Rost '81)

Conjecture: g differentiable, strictly concave, and satisfies the curvature assumption.

FPP: some non-lattice models with rotational symmetry
Euclidean FPP: Howard & Newman '97, '01

LPP: $\omega_x \sim \exp(1)$ gives $g(\xi) = (\sqrt{\xi_1} + \sqrt{\xi_2})^2$ (Rost '81)

Ferrari-Pimentel '05, Cator-Pimentel '12

Conjecture: g differentiable, strictly concave, and satisfies the curvature assumption.

FPP: some non-lattice models with rotational symmetry
Euclidean FPP: Howard & Newman '97, '01

LPP: $\omega_x \sim \exp(1)$ gives $g(\xi) = (\sqrt{\xi_1} + \sqrt{\xi_2})^2$ (Rost '81)

Ferrari-Pimentel '05, Cator-Pimentel '12

Burgers equation with kick forcing: Bakhtin-Cator-Khanin '14
Bakhtin '16

Start with Busemann functions as the fundamental object
to avoid a priori assumptions on the shape

Start with Busemann functions as the fundamental object to avoid a priori assumptions on the shape

Damron-Hanson '14 & '17 (FPP with general weights)
Georgiou-RA-Seppäläinen '17 (LPP with general weights)

Start with Busemann functions as the fundamental object to avoid a priori assumptions on the shape

Damron-Hanson '14 & '17 (FPP with general weights)
Georgiou-RA-Seppäläinen '17 (LPP with general weights)

Define B^{ξ} through weak subsequential limits of B^h_n with $h = -\nabla g(\xi)$

Start with Busemann functions as the fundamental object to avoid a priori assumptions on the shape

Damron-Hanson '14 & '17 (FPP with general weights)
Georgiou-RA-Seppäläinen '17 (LPP with general weights)

Define B^{ξ} through weak subsequential limits of B^h_n with $h=-\nabla g(\xi)$

Do this for all $\xi \in D_0$, a countable dense set of diff. points of g

Start with Busemann functions as the fundamental object to avoid a priori assumptions on the shape

Damron-Hanson '14 & '17 (FPP with general weights)
Georgiou-RA-Seppäläinen '17 (LPP with general weights)

Define B^{ξ} through weak subsequential limits of B^{h}_{n} with $h=-\nabla g(\xi)$

Do this for all $\xi \in D_0$, a countable dense set of diff. points of g

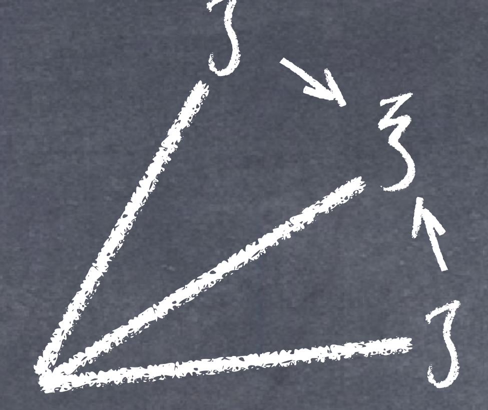
Cocycle, recovery, monotonicity hold

Monotonicity implies $\quad B^{\zeta\pm}(x,x+e_i)=\lim\limits_{D_0 \ni \eta \to \zeta} B^{\eta}(x,x+e_i)$

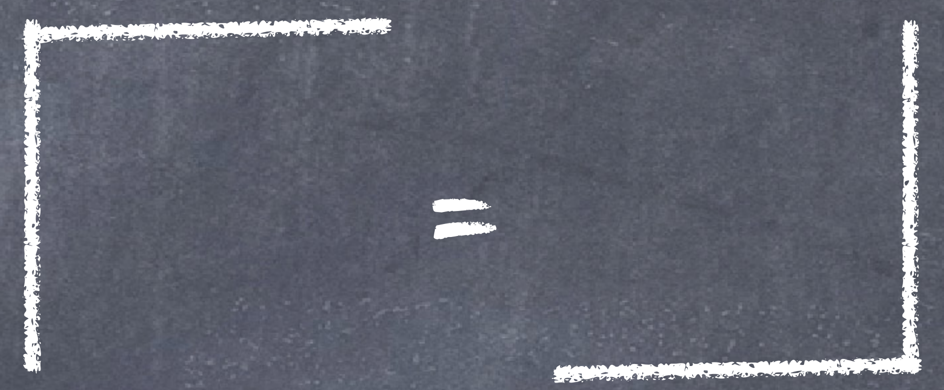
Monotonicity implies $B^{\xi\pm}(x,x+e_i)=\lim_{\mathcal{D}_0\ni\zeta\to\xi} B^{\zeta}(x,x+e_i)$

Cocycle property extends this to $B^{\xi\pm}(x,y)$

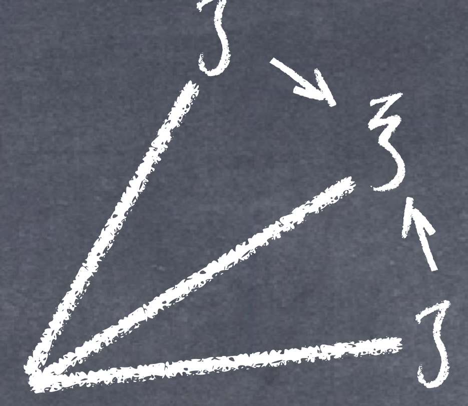
Monotonicity implies $B^{\xi\pm}(x,x+e_i)=\lim_{D\ni\zeta\to\xi} B^{\zeta}(x,x+e_i)$



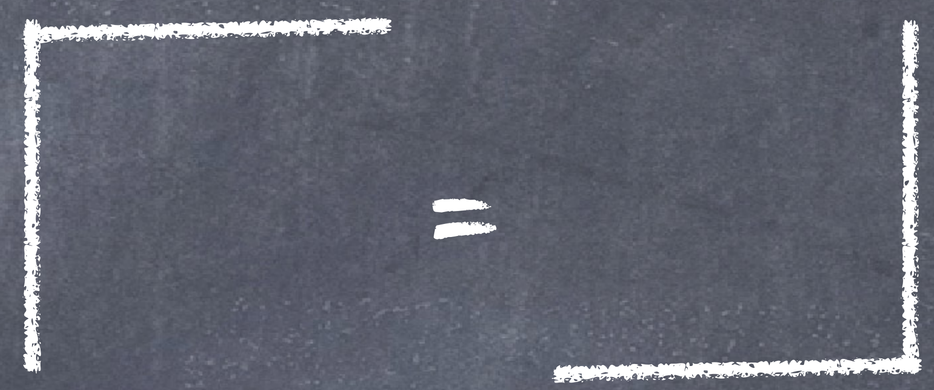
Cocycle property extends this to $B^{\xi\pm}(x,y)$

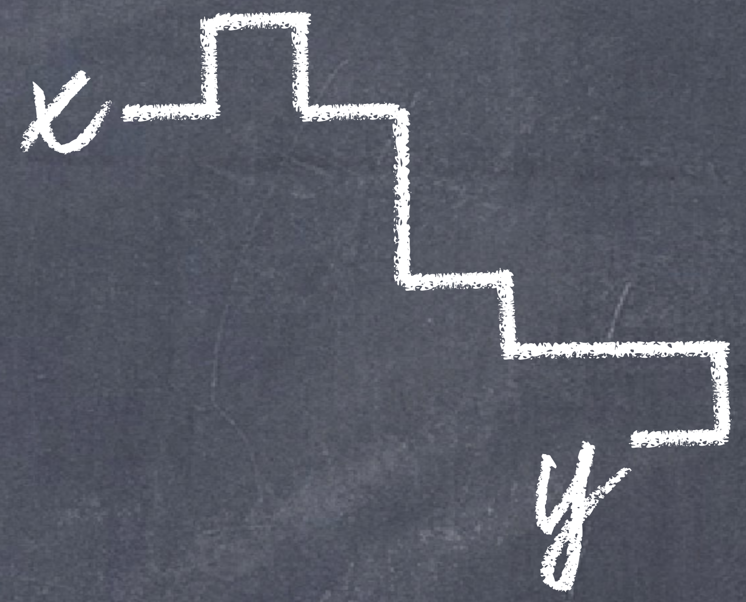


Monotonicity implies $B^{\xi\pm}(x,x+e_i)=\lim_{D\ni\zeta\to\xi} B^{\zeta}(x,x+e_i)$

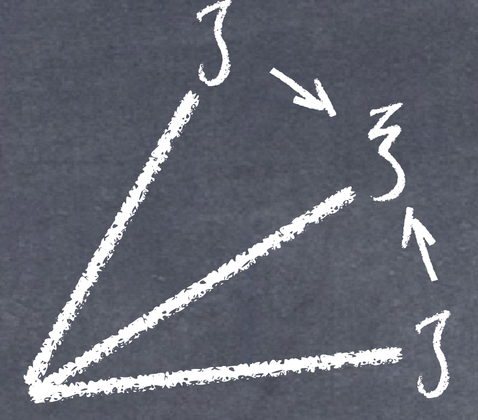


Cocycle property extends this to $B^{\xi\pm}(x,y)$

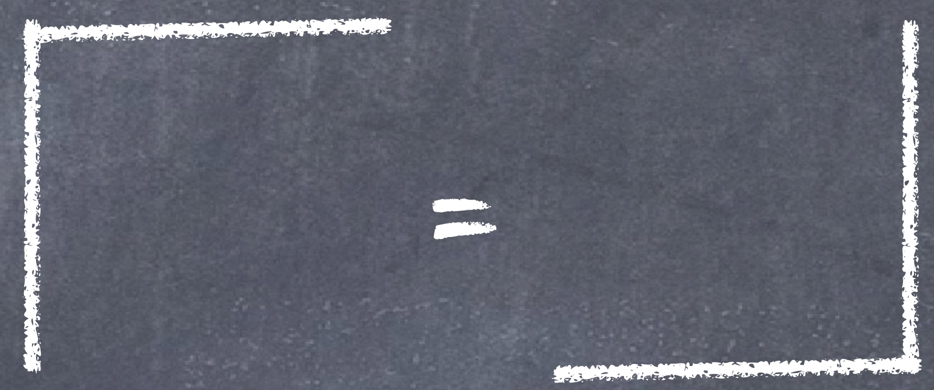




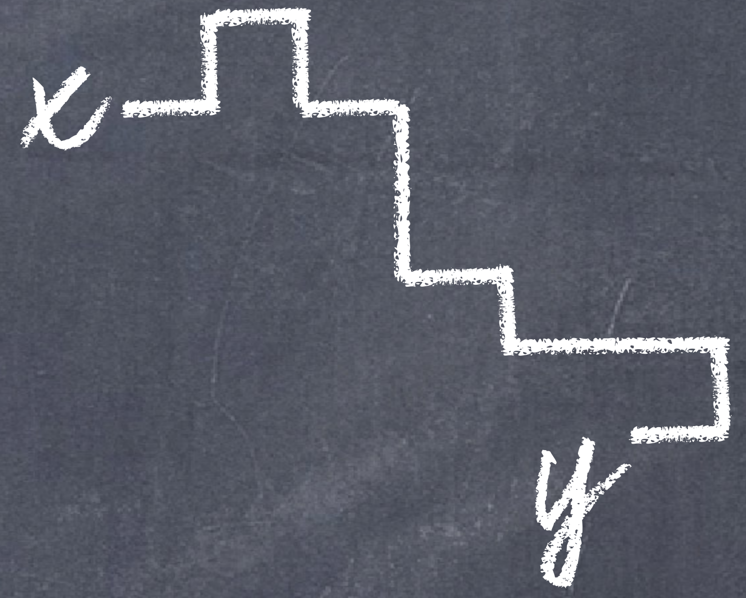
Monotonicity implies $B^{\xi\pm}(x,x+e_i)=\lim\limits_{\mathcal{D}_0\ni\zeta\to\xi} B^{\zeta}(x,x+e_i)$



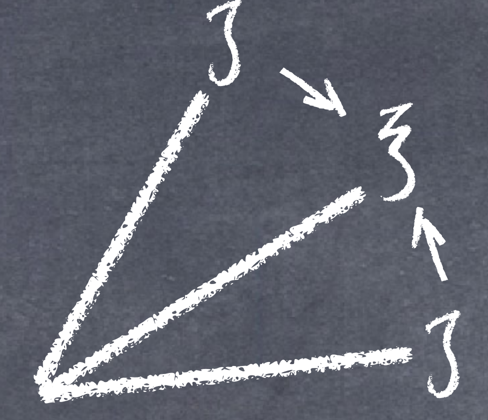
Cocycle property extends this to $B^{\xi\pm}(x,y)$



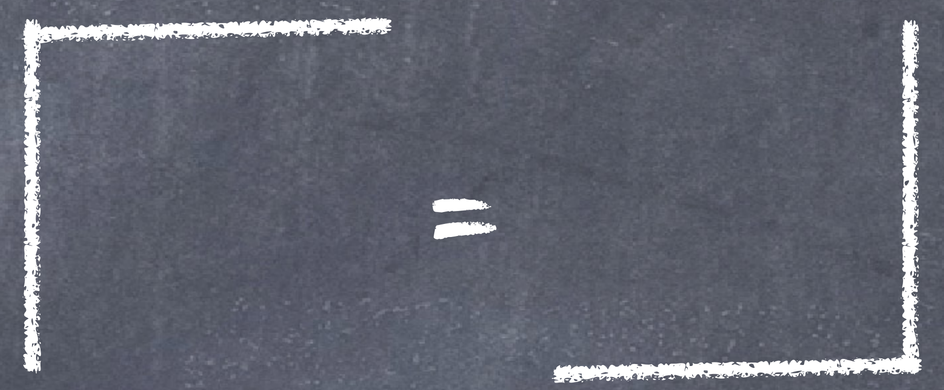
Cocycle, recovery, monotonicity still hold



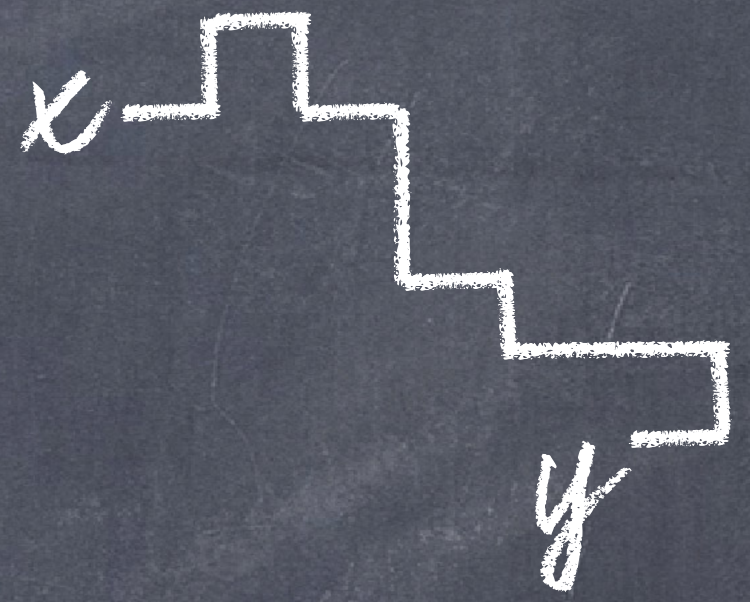
Monotonicity implies $B^{\xi\pm}(x,x+e_i)=\lim\limits_{D_0\ni\zeta\to\xi} B^{\zeta}(x,x+e_i)$



Cocycle property extends this to $B^{\xi\pm}(x,y)$

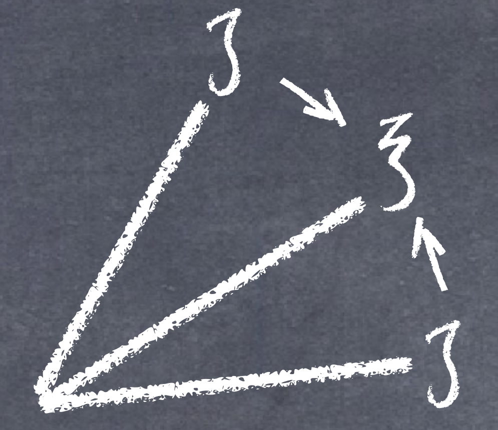


Cocycle, recovery, monotonicity still hold

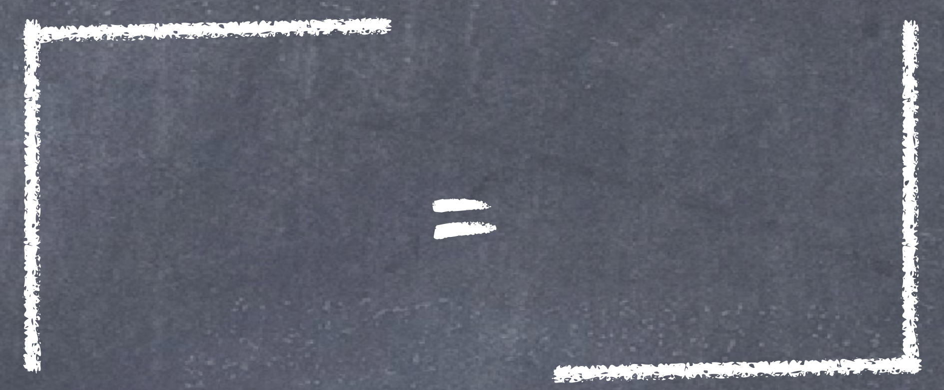


Consistency: $B^{\xi\pm}=B^{\xi}$ when $\xi\in D_0$

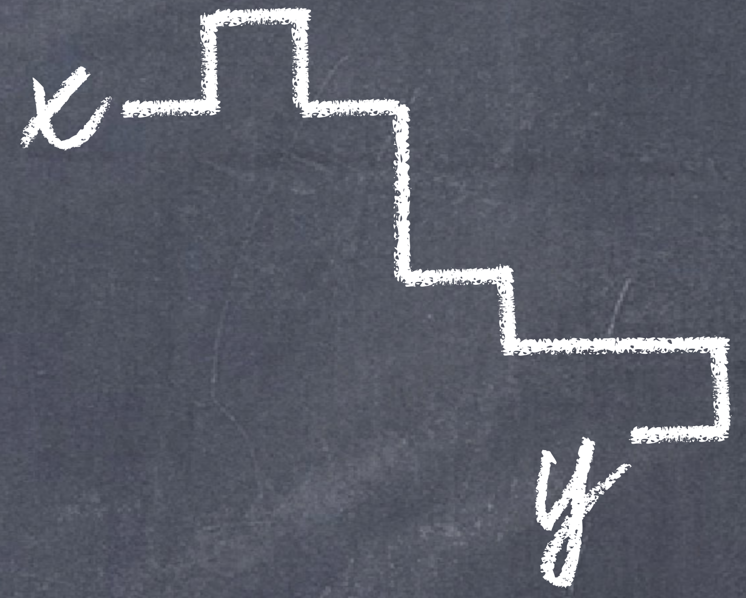
Monotonicity implies $B^{\xi\pm}(x,x+e_i)=\lim\limits_{D_0\ni\zeta\to\xi} B^{\zeta}(x,x+e_i)$



Cocycle property extends this to $B^{\xi\pm}(x,y)$



Cocycle, recovery, monotonicity still hold

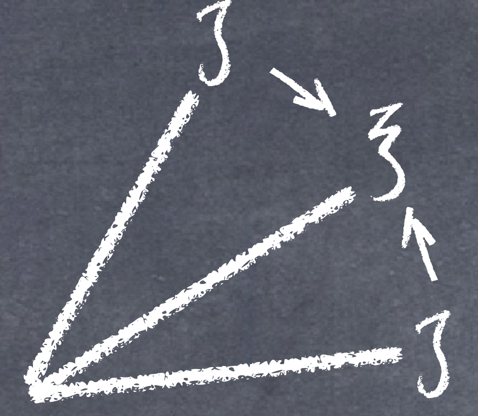


Consistency: $B^{\xi\pm}=B^{\xi}$ when $\xi\in D_0$

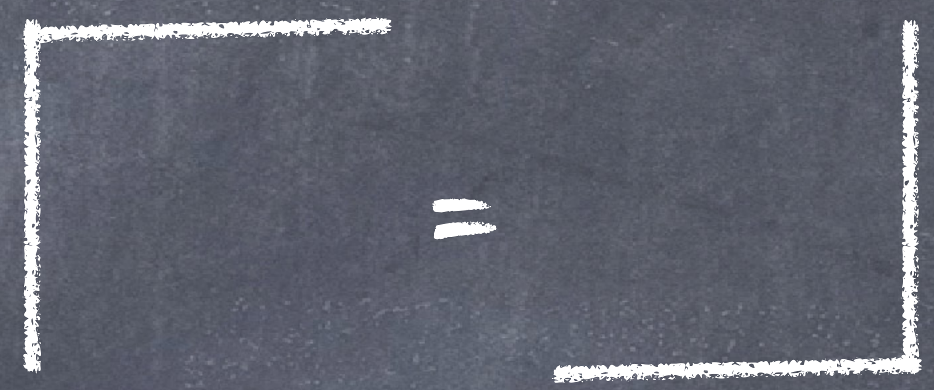
Monotonicity implies $G_{x,x_n}-G_{y,x_n}\to B^{\xi+}(x,y)=B^{\xi-}(x,y)$ a.s.

$G_{x(n)}(h)-G_{y(n)}(h) \nearrow$

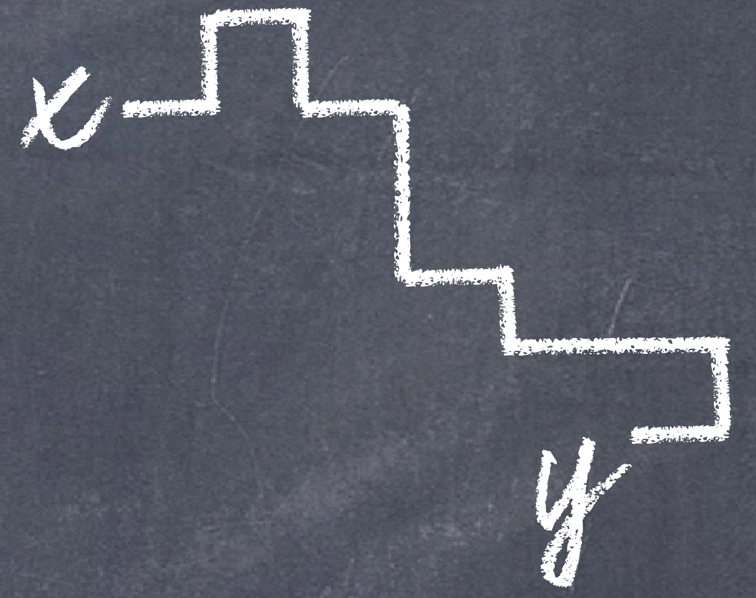
Monotonicity implies $B^{\xi\pm}(x,x+e_i)=\lim\limits_{\mathcal{D}_0\ni\zeta\to\xi} B^{\zeta}(x,x+e_i)$



Cocycle property extends this to $B^{\xi\pm}(x,y)$



Cocycle, recovery, monotonicity still hold



Consistency: $B^{\xi\pm}=B^{\xi}$ when $\xi\in\mathcal{D}_0$

Monotonicity implies $G_{x,x_n}-G_{y,x_n}\to B^{\xi+}(x,y)=B^{\xi-}(x,y)$ a.s.

$G_{x(n)}(h)-G_{y(n)}(h)$

(for example) holds for all ξ if g is everywhere differentiable

$\gamma^{x,\xi\pm}$: start at x and follow minimal $B^{\xi\pm}$-increment

$\gamma^{x,\xi\pm}$: start at x and follow minimal $B^{\xi\pm}$-increment

cocycle+recovery imply that $\gamma^{x,\xi\pm}$ are geodesics

$\gamma^{x,\xi\pm}$: start at x and follow minimal $B^{\xi\pm}$-increment

cocycle+recovery imply that $\gamma^{x,\xi\pm}$ are geodesics

<u>Theorem:</u> a.s. $\forall x,\xi,\square\in\{-,+\}$, $\gamma^{x,\xi\square}$ is directed into $\underbrace{\{\zeta : \nabla g(\zeta)=\nabla g(\xi\square)\}}_{\mathcal{U}_{\xi\square}}$

$\gamma^{x,\xi\pm}$: start at x and follow minimal $B^{\xi\pm}$-increment

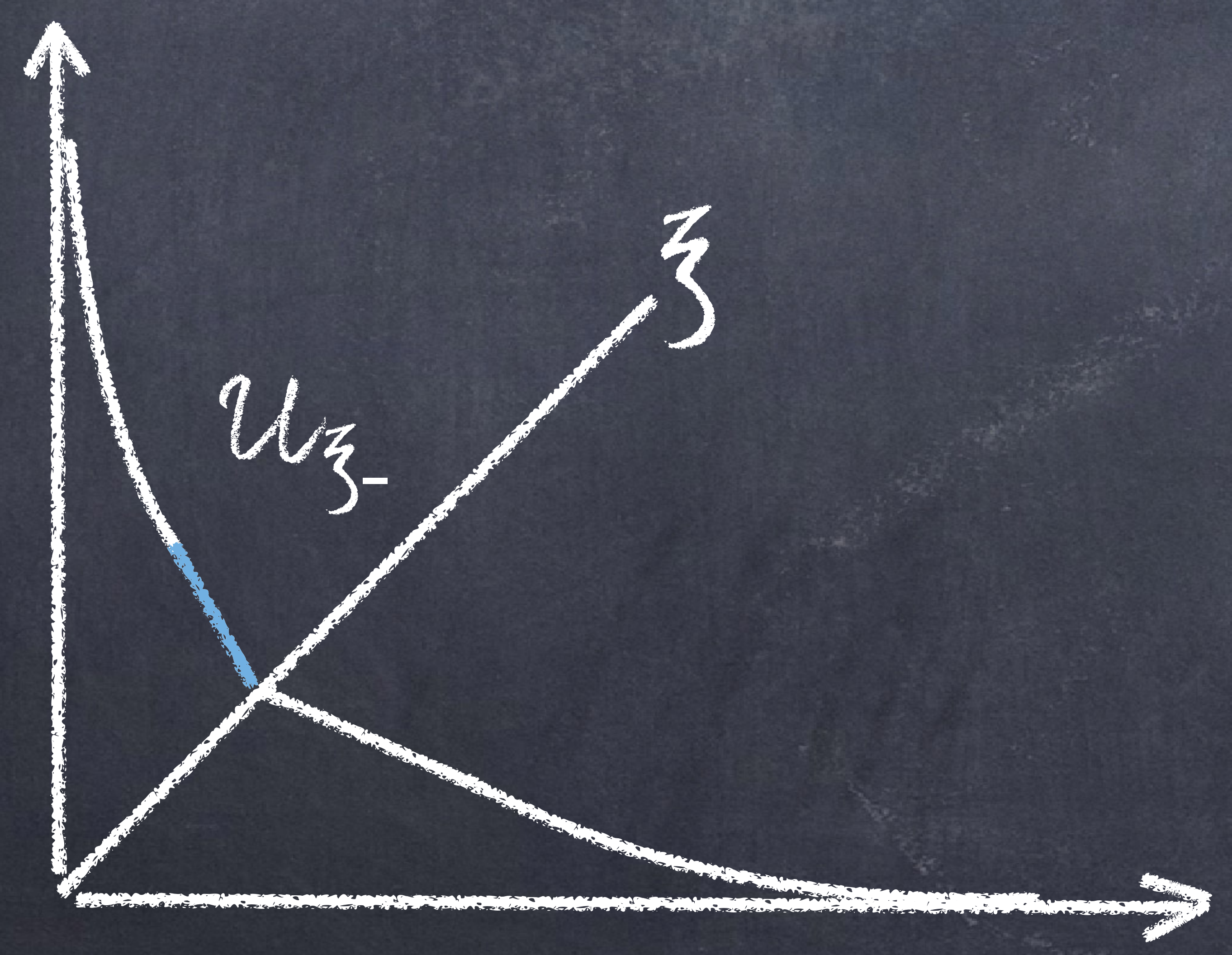
cocycle+recovery imply that $\gamma^{x,\xi\pm}$ are geodesics

<u>Theorem:</u> a.s. $\forall x, \xi, \square \in \{-,+\}$, $\gamma^{x,\xi\square}$ is directed into $\underbrace{\{\zeta : \nabla g(\zeta) = \nabla g(\xi\square)\}}_{\mathcal{U}_{\xi\square}}$

$\gamma^{x,\xi\pm}$: start at x and follow minimal $B^{\xi\pm}$-increment

cocycle+recovery imply that $\gamma^{x,\xi\pm}$ are geodesics

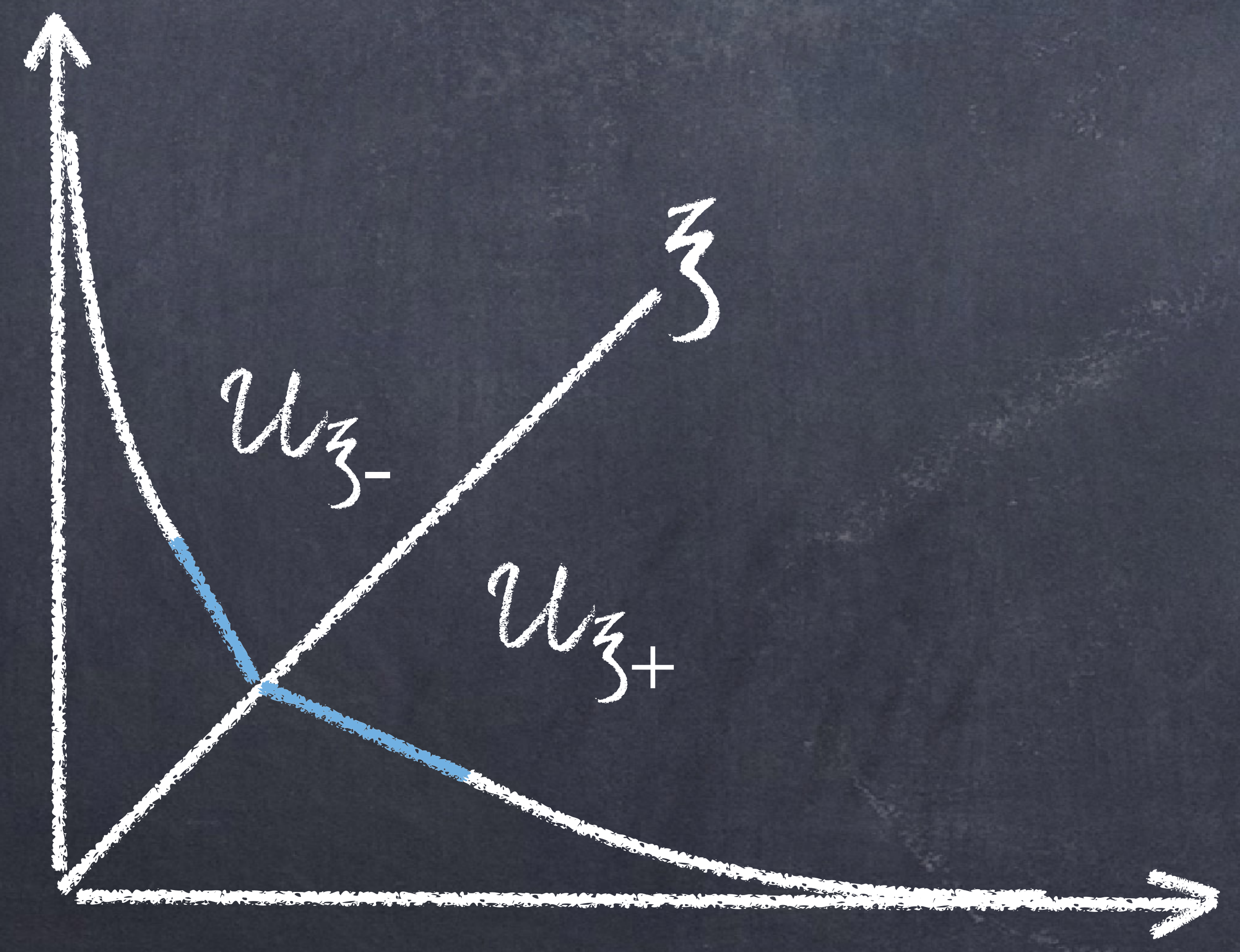
Theorem: a.s. $\forall x, \xi, \square \in \{-,+\}$, $\gamma^{x,\xi\square}$ is directed into $\underbrace{\{\zeta : \nabla g(\zeta) = \nabla g(\xi\square)\}}_{\mathcal{U}_{\xi\square}}$

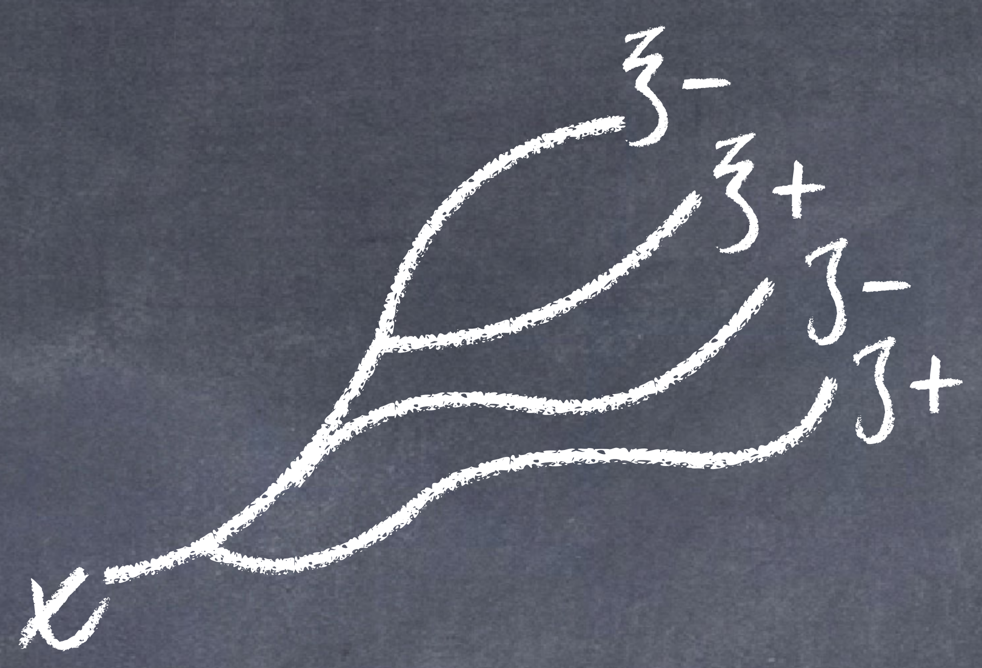


$\gamma^{x,\xi\pm}$: start at x and follow minimal $B^{\xi\pm}$-increment

cocycle+recovery imply that $\gamma^{x,\xi\pm}$ are geodesics

<u>Theorem:</u> a.s. $\forall x,\xi,\square\in\{-,+\}$, $\gamma^{x,\xi\square}$ is directed into $\underbrace{\{\zeta : \nabla g(\zeta)=\nabla g(\xi\square)\}}_{\mathcal{U}_{\xi\square}}$



Monotonicity implies 

if 

Monotonicity implies

if

Then we can trap any geodesic between $\gamma^{x,\xi\pm}$ for some ξ

Monotonicity implies 

if 

Then we can trap any geodesic between $\gamma^{x,\xi\pm}$ for some ξ

<u>Corollary</u>: a.s. all geodesics are directed into $\mathcal{U}_{\xi-} \cup \mathcal{U}_{\xi+}$ for some ξ

Monotonicity implies

if

Then we can trap any geodesic between $\gamma^{x,\xi\pm}$ for some ξ

<u>Corollary:</u> a.s. all geodesics are directed into $\mathcal{U}_{\xi-} \cup \mathcal{U}_{\xi+}$ for some ξ

<u>Corollary:</u> If g is strictly concave then a.s. all geo. are ξ-directed for some ξ

Monotonicity implies 

if 

Then we can trap any geodesic between $\gamma^{x,\xi\pm}$ for some ξ

<u>Corollary:</u> a.s. all geodesics are directed into $\mathcal{U}_{\xi-} \cup \mathcal{U}_{\xi+}$ for some ξ

<u>Corollary:</u> If g is strictly concave then a.s. all geo. are ξ-directed for some ξ

If g is differentiable then a.s. all geodesics are directed into a linear segment

Theorem: $\forall\, \xi, x, y, \square \in \{-,+\}$: $\gamma^{x,\xi\square}$ and $\gamma^{y,\xi\square}$ a.s. coalesce

<u>Theorem:</u> $\forall \xi, x, y, \square \in \{-,+\}$: $\gamma^{x,\xi\square}$ and $\gamma^{y,\xi\square}$ a.s. coalesce

<u>Theorem:</u> g differentiable implies $\forall \xi, x$: $\gamma^{x,\xi+} = \gamma^{x,\xi-}$ a.s.

Theorem: $\forall \xi, x, y, \square \in \{-,+\}$: $\gamma^{x,\xi\square}$ and $\gamma^{y,\xi\square}$ a.s. coalesce

Theorem: g differentiable implies $\forall \xi, x$: $\gamma^{x,\xi+} = \gamma^{x,\xi-}$ a.s.

Corollary: g diff. implies $\forall \xi, x$ a.s. $\exists!$ $\mathcal{U}_\xi$-directed geo. out of x

__Theorem:__ $\forall \xi, x, y, \square \in \{-,+\}$: $\gamma^{x,\xi\square}$ and $\gamma^{y,\xi\square}$ a.s. coalesce

__Theorem:__ g differentiable implies $\forall \xi, x$: $\gamma^{x,\xi+} = \gamma^{x,\xi-}$ a.s.

__Corollary:__ g diff. implies $\forall \xi, x$ a.s. $\exists!$ $\mathcal{U}_\xi$-directed geo. out of x

__But__ Null set depends on ξ

Theorem: $\forall \xi, x, y, \Box \in \{-,+\}$: $\gamma^{x,\xi\Box}$ and $\gamma^{y,\xi\Box}$ a.s. coalesce

Theorem: g differentiable implies $\forall \xi, x$: $\gamma^{x,\xi+} = \gamma^{x,\xi-}$ a.s.

Corollary: g diff. implies $\forall \xi, x$ a.s. $\exists!$ $\mathcal{U}_\xi$-directed geo. out of x

But Null set depends on ξ

Uniqueness and coalescence for all ξ?

Theorem: $\forall \xi, x, y, \square \in \{-,+\}$: $\gamma^{x,\xi\square}$ and $\gamma^{y,\xi\square}$ a.s. coalesce

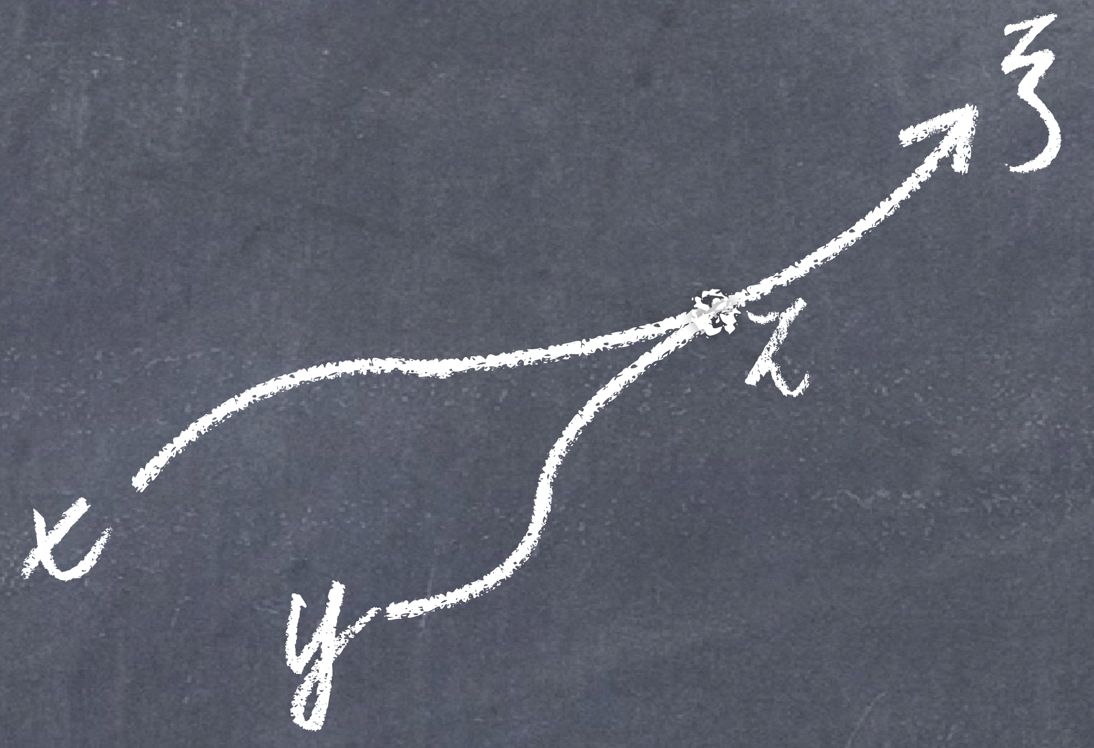
Theorem: g differentiable implies $\forall \xi, x$: $\gamma^{x,\xi+} = \gamma^{x,\xi-}$ a.s.

Corollary: g diff. implies $\forall \xi, x$ a.s. $\exists!$ $\mathcal{U}_\xi$-directed geo. out of x

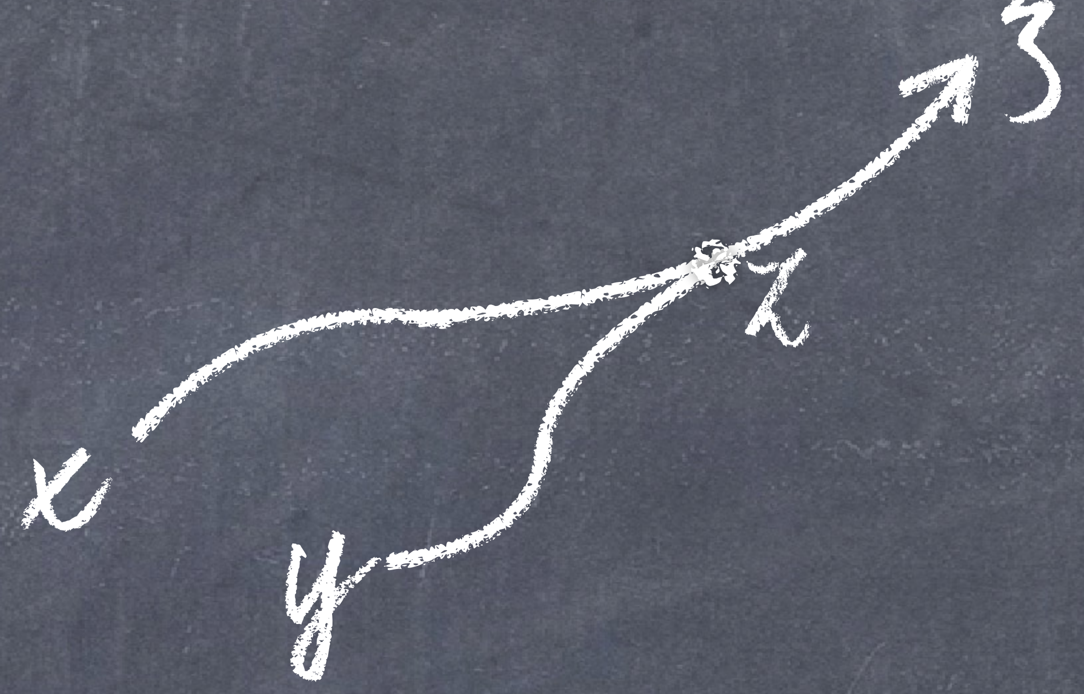
But Null set depends on ξ

Uniqueness and coalescence for all ξ?

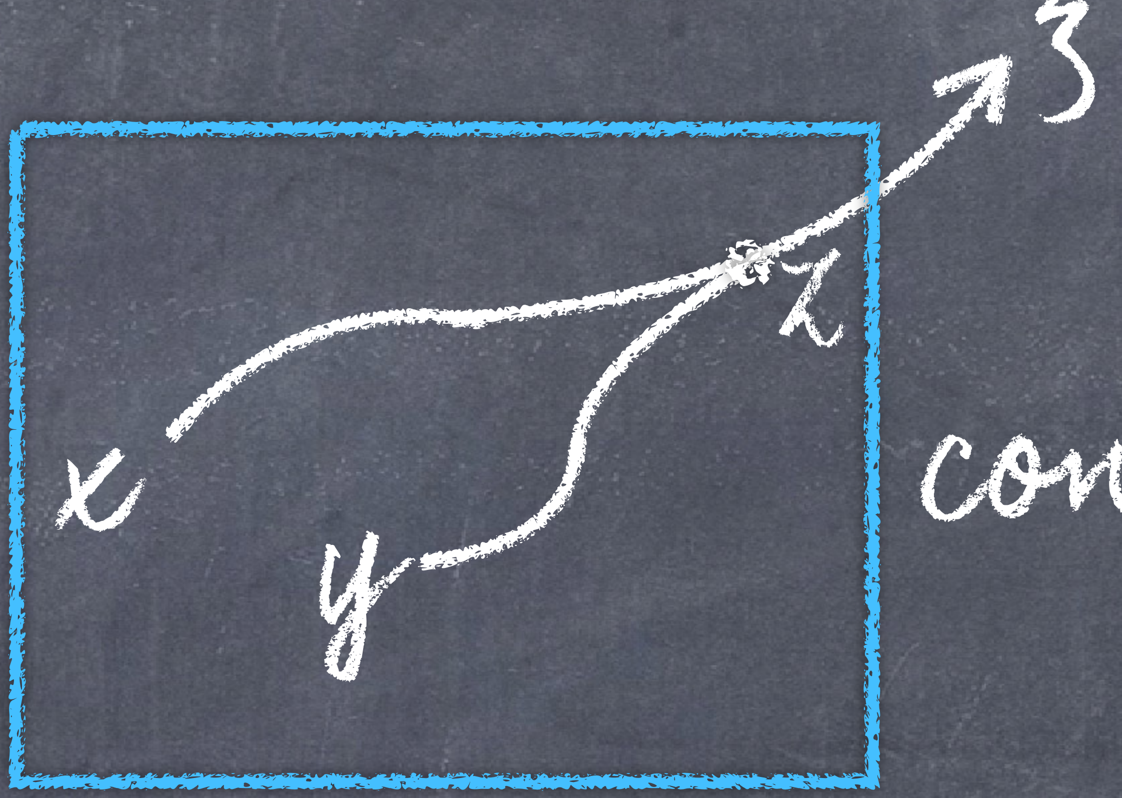
I.e. want to switch "$\forall \xi$ for a.e. ω" to "for a.e. ω $\forall \xi$"


ξ
z
x
y

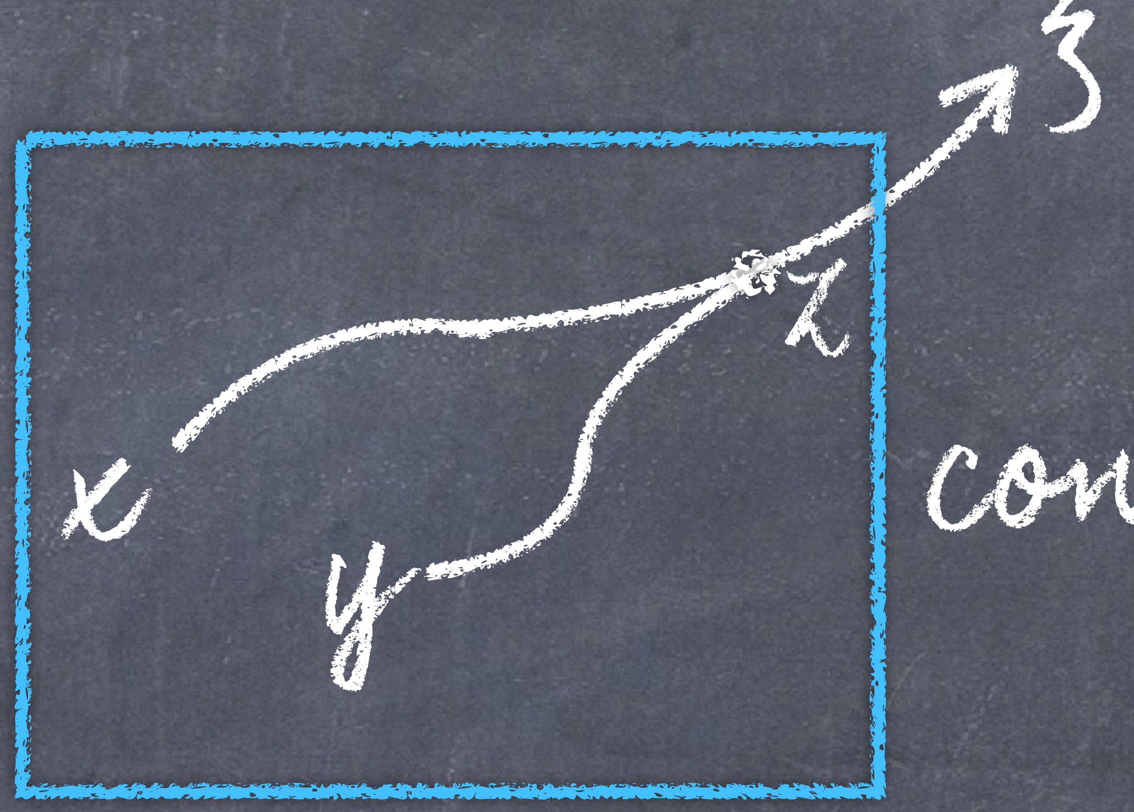
$B^{\xi}(x,y) = G_{x,z} - G_{y,z}$



$B^{\xi}(x,y) = G_{x,z} - G_{y,z}$

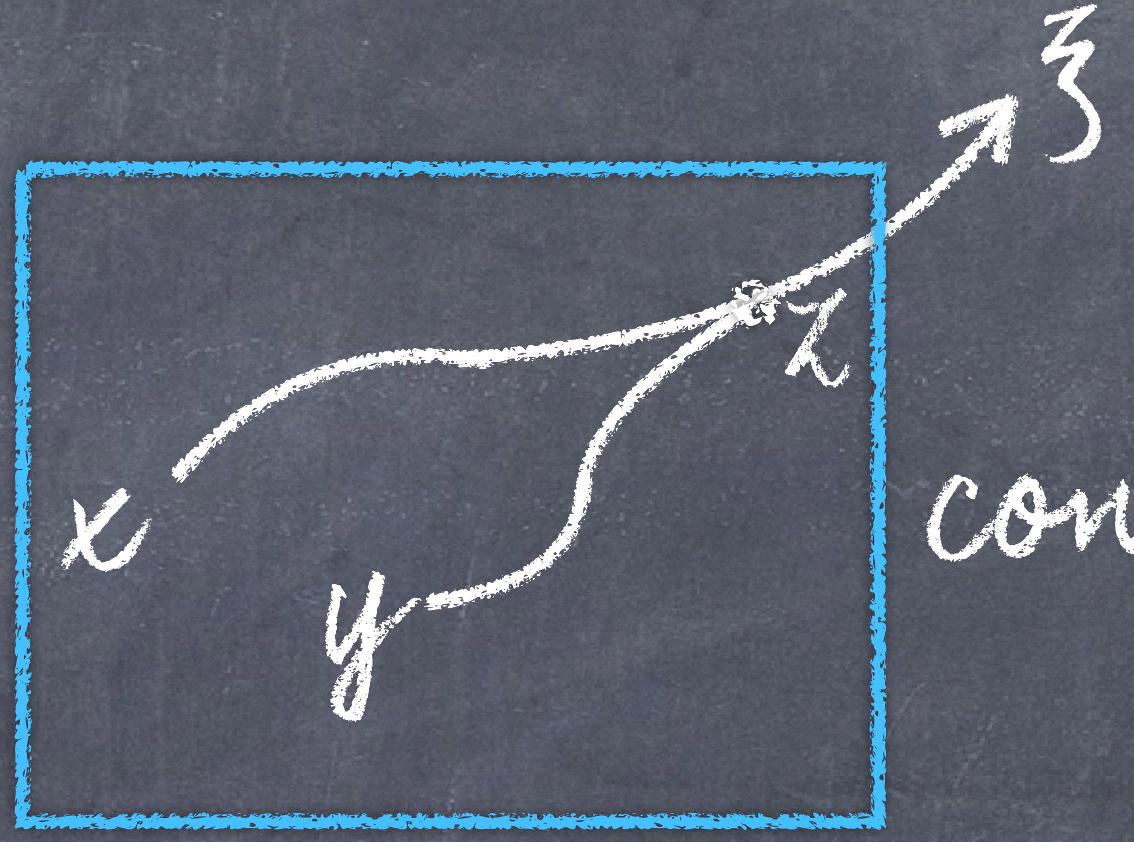
$B^{\xi}(x,y)$ constant over an interval of ξ's implies  constant

$B^{\xi}(x,y) = G_{x,z} - G_{y,z}$

$B^{\xi}(x,y)$ constant over an interval of ξ's implies  constant

$\mu_{x,y}$: Lebesgue-Stieltjes signed measure of $\xi \to B^{\xi}(x,y)$

$B^{\xi}(x,y)=G_{x,z}-G_{y,z}$

$B^{\xi}(x,y)$ constant over an interval of ξ's implies 

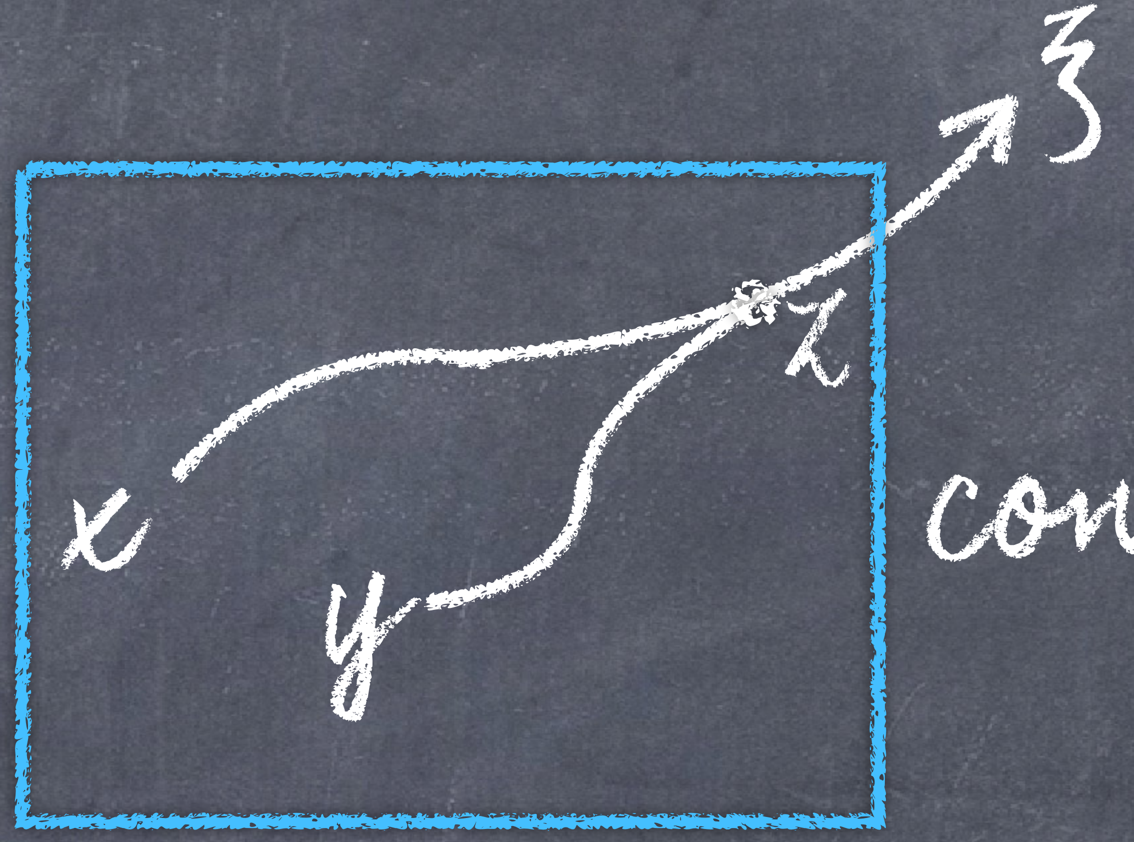
constant

$\mu_{x,y}$: Lebesgue-Stieltjes signed measure of $\xi \to B^{\xi}(x,y)$

Set of exceptional directions $\mathcal{V}^{\omega}=\bigcup_{x,y} \operatorname{supp} \mu_{x,y}$

$B^{\xi}(x,y) = G_{x,z} - G_{y,z}$

$B^{\xi}(x,y)$ constant over an interval of ξ's implies



constant

$\mu_{x,y}$: Lebesgue-Stieltjes signed measure of $\xi \to B^{\xi}(x,y)$

Set of exceptional directions $\mathcal{V}^{\omega} = \bigcup_{x,y} \operatorname{supp} \mu_{x,y}$

<u>Theorem:</u> $\xi \notin \mathcal{V}^{\omega} \iff \gamma^{x,\xi+} = \gamma^{x,\xi-} = \gamma^{x,\xi}$, $\gamma^{x,\xi}$ & $\gamma^{y,\xi}$ coalesce

__Nice case:__ g is differentiable at endpoints of its linear segments (e.g. differentiable or strictly concave)

<u>Nice case:</u> g is differentiable at endpoints of its linear segments (*)
(e.g. differentiable or strictly concave)

__Nice case:__ g is differentiable at endpoints of its linear segments (*)
(e.g. differentiable or strictly concave)

__Theorem:__ Under (*), a.s. uniqueness & coalescence of $\mathcal{U}_\xi$-directed geodesics for all $\xi \notin \mathcal{V}^\omega$

Nice case: g is differentiable at endpoints of its linear segments (*)
(e.g. differentiable or strictly concave)

Theorem: Under (*), a.s. uniqueness & coalescence of $\mathcal{U}_\xi$-directed geodesics for all $\xi \notin \mathcal{V}^\omega$

Size and structure of $\mathcal{V}^\omega$

<u>Nice case:</u> g is differentiable at endpoints of its linear segments (*)
(e.g. differentiable or strictly concave)

<u>Theorem:</u> Under (*), a.s. uniqueness & coalescence of $\mathcal{U}_\xi$-directed geodesics for all $\xi \notin \mathcal{V}^\omega$

<u>Size and structure of $\mathcal{V}^\omega$</u>

$\mathcal{V}^\omega$ is dense away from linear segments of g

Nice case: g is differentiable at endpoints of its linear segments (*)
(e.g. differentiable or strictly concave)

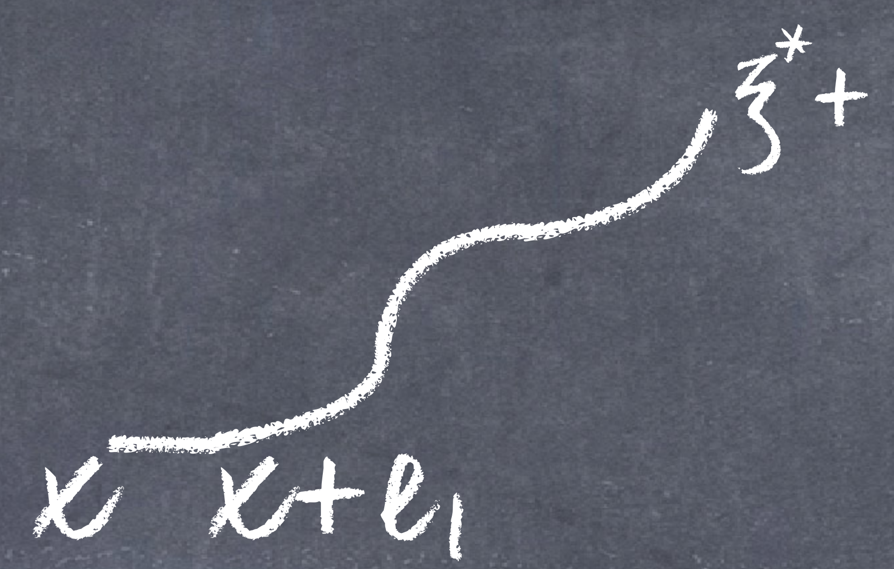
Theorem: Under (*), a.s. uniqueness & coalescence of $\mathcal{U}_\xi$-directed geodesics for all $\xi \notin \mathcal{V}^\omega$

Size and structure of $\mathcal{V}^\omega$

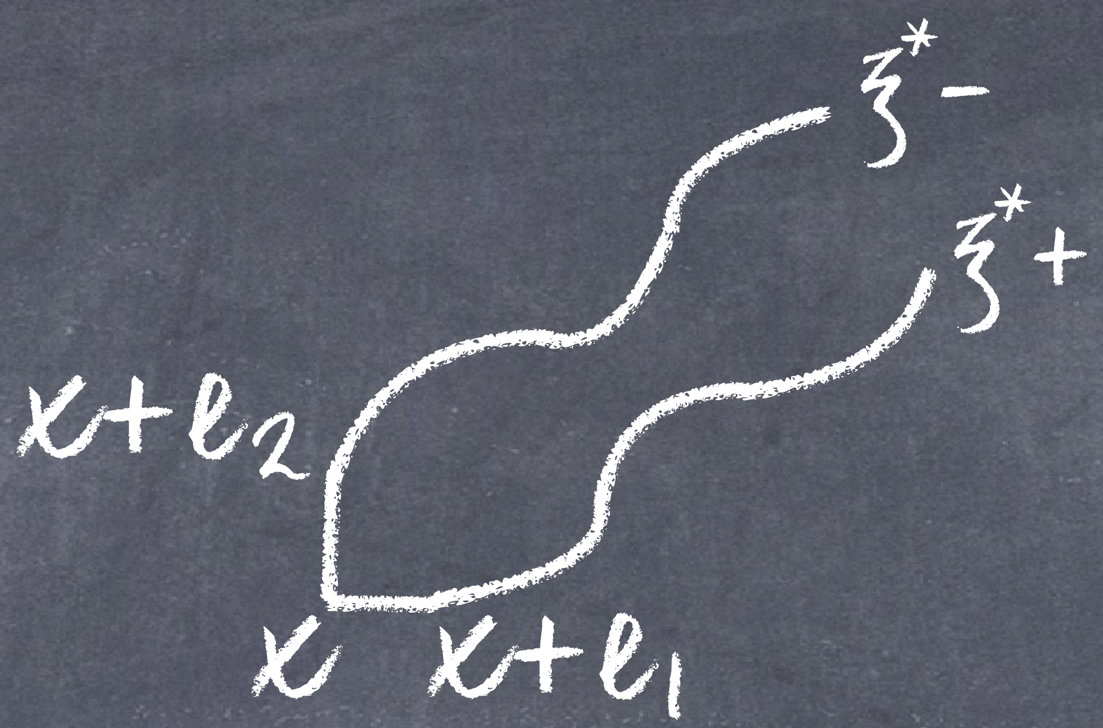
$\mathcal{V}^\omega$ is dense away from linear segments of g

Complement is a countable intersection of dense open sets

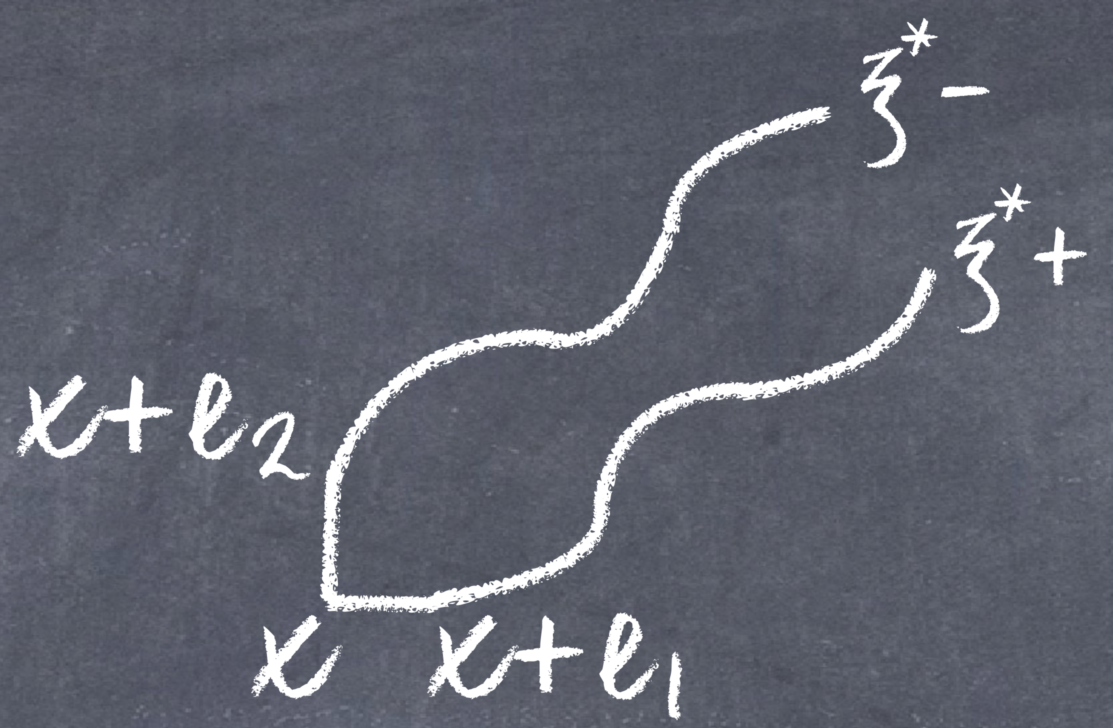
Special members: $\xi^*(x)$=unique direction s.t.



Special members: $\xi^*(x)$=unique direction s.t.

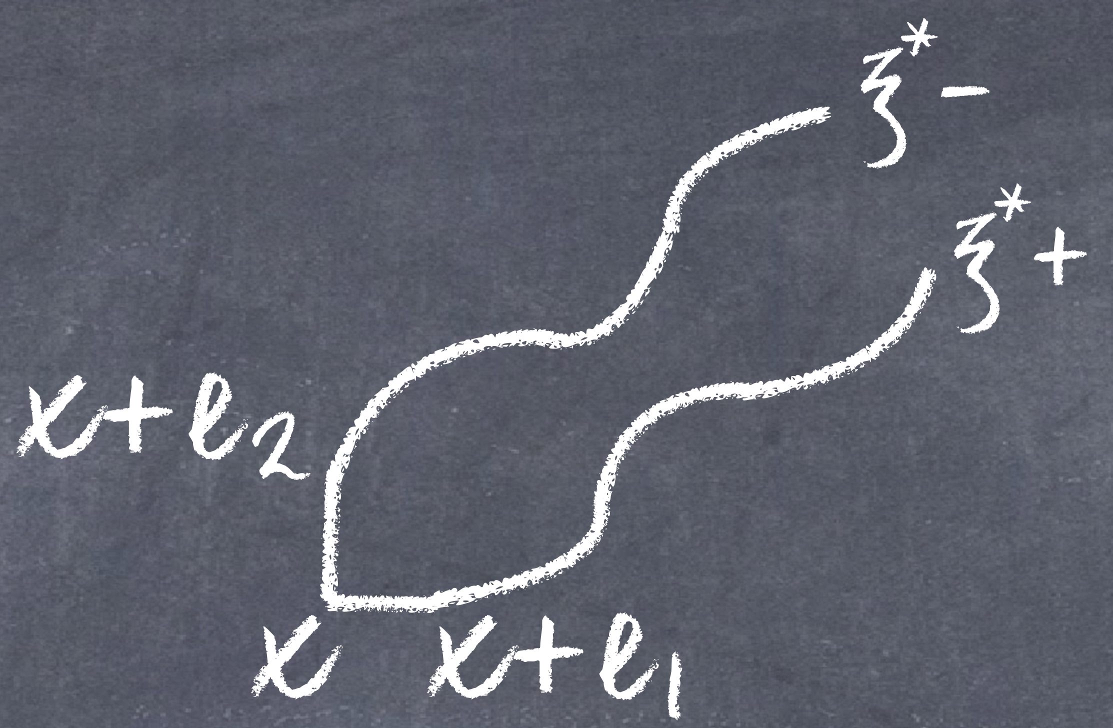


Special members: $\xi^*(x)$=unique direction s.t.



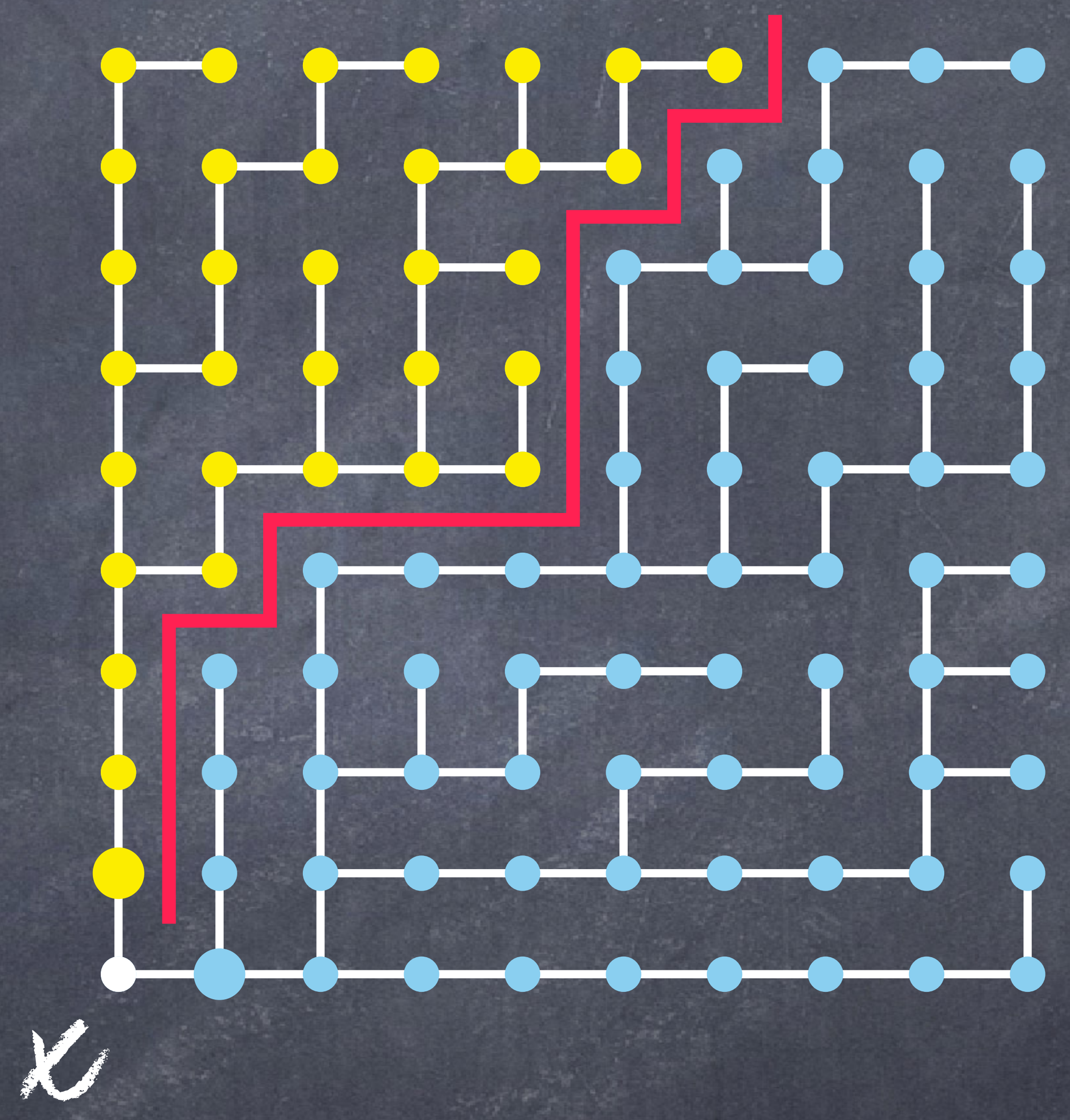
This is why deterministic ξ is not enough

Special members: $\xi^*(x)$=unique direction s.t.

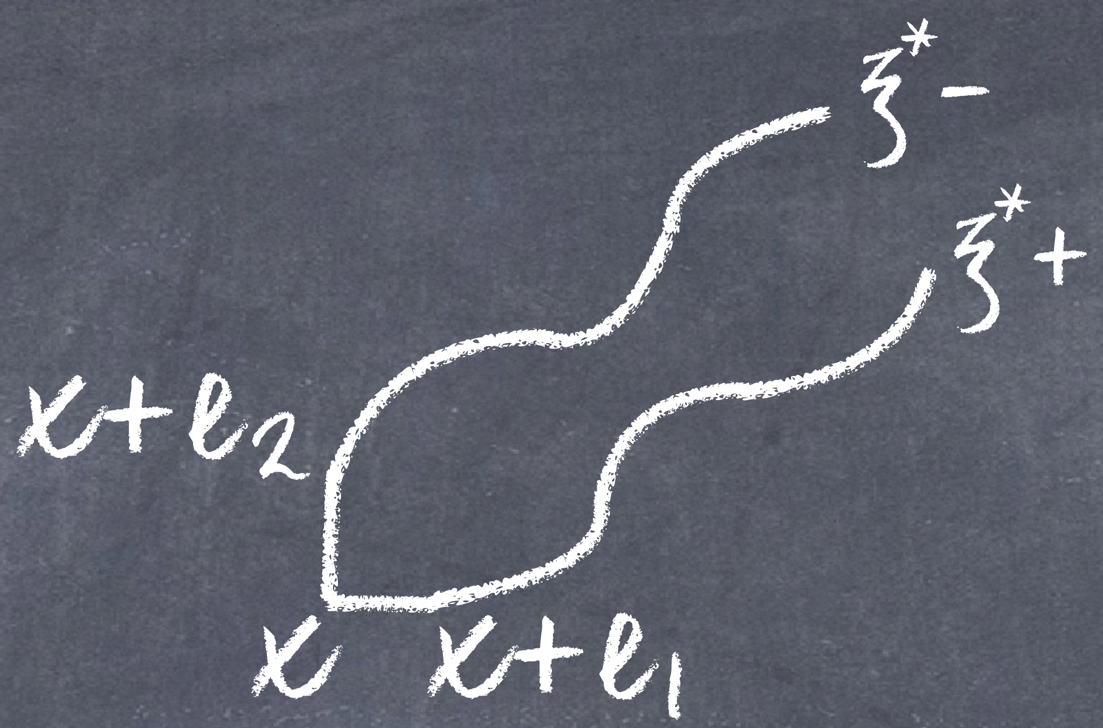


This is why deterministic ξ is not enough

Tree of geodesics from x:

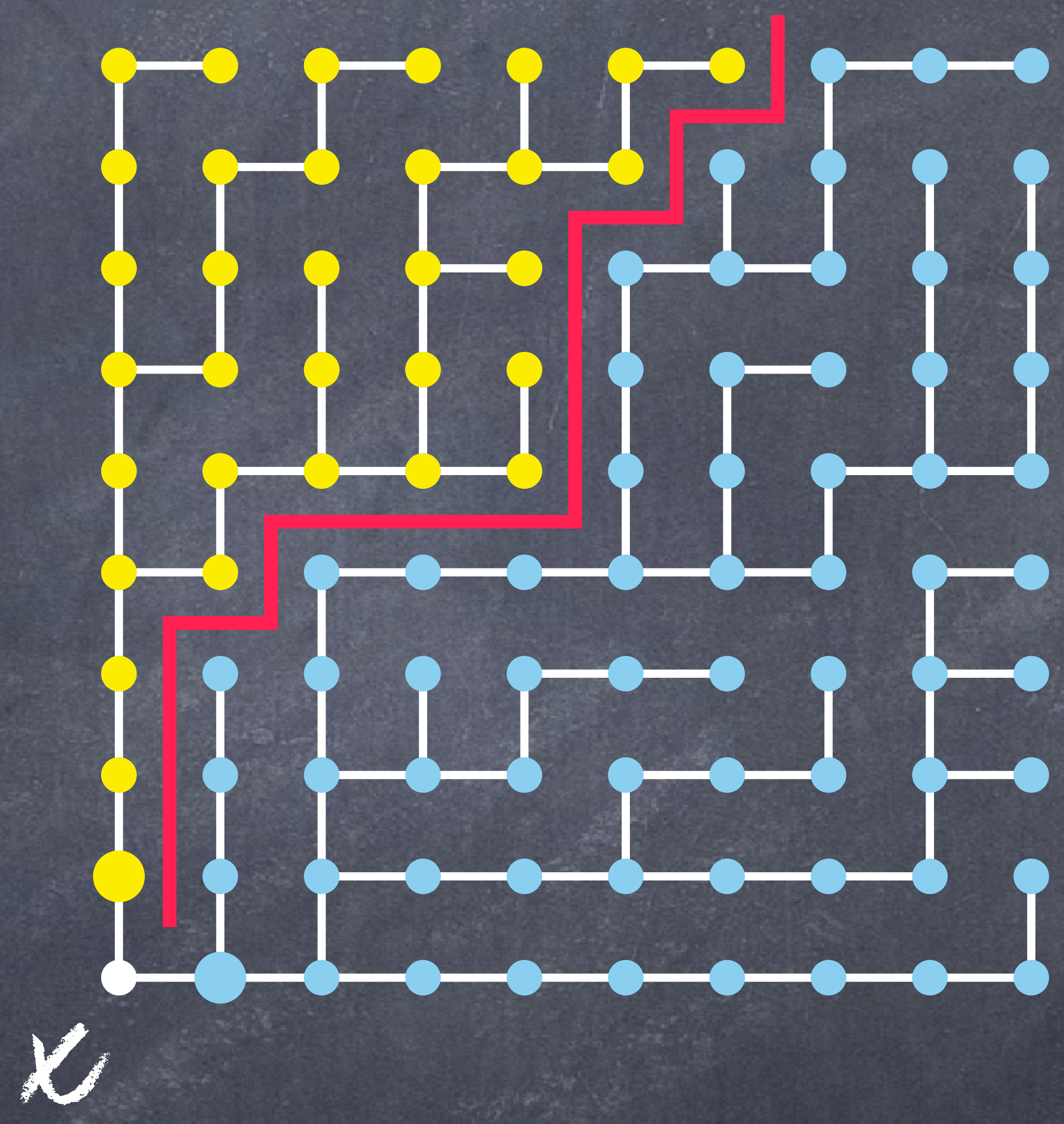


Special members: $\xi^*(x)$=unique direction s.t.



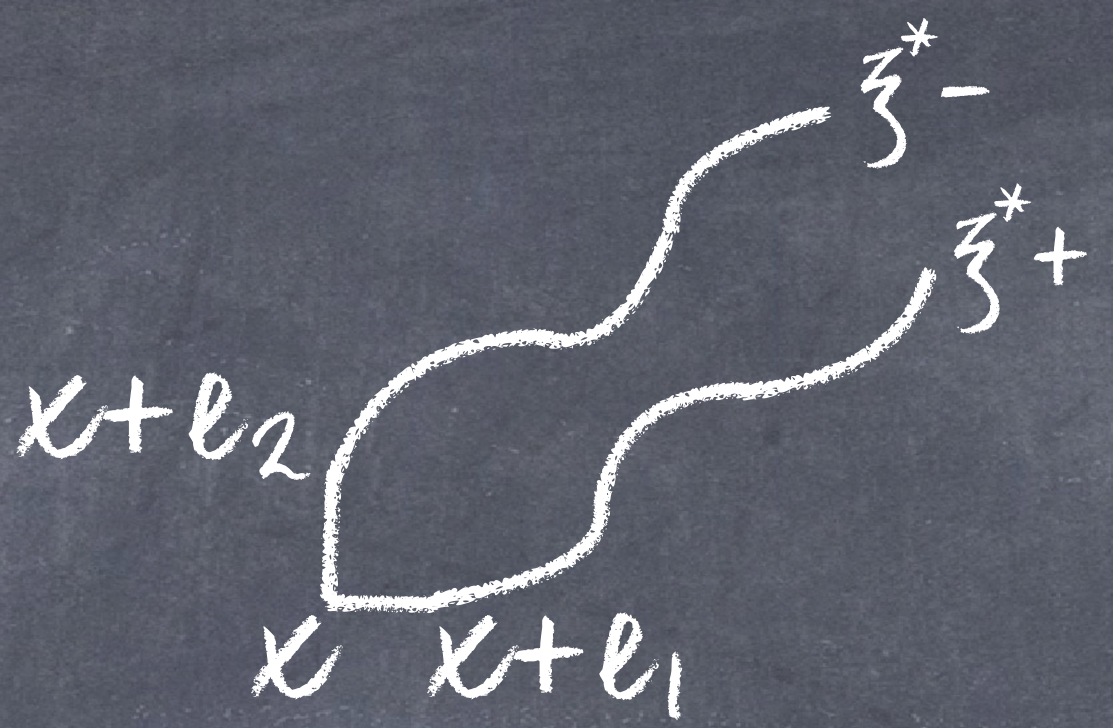
This is why deterministic ξ is not enough

Tree of geodesics from x:



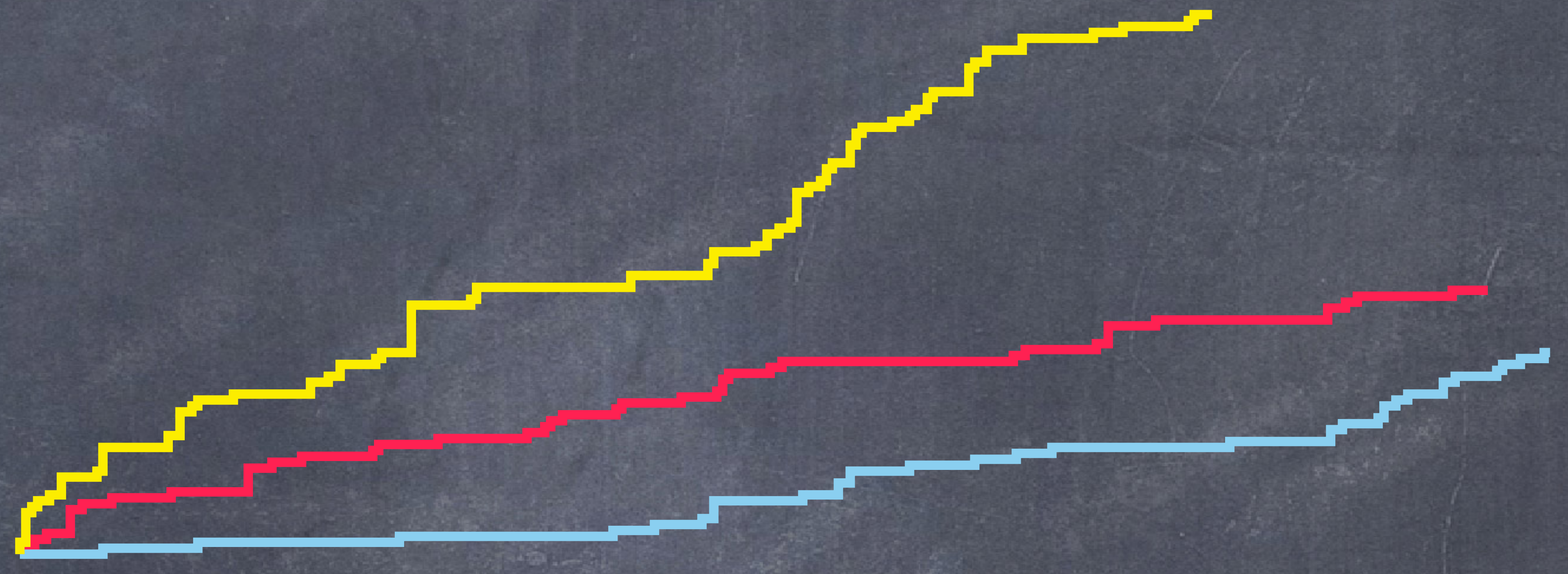
Under $(*)$: $\xi^*(x)$ is the asymptotic direction of the competition interface between subtrees rooted at $x+e_1$ and $x+e_2$

Special members: $\xi^*(x)$=unique direction s.t.



This is why deterministic ξ is not enough

Tree of geodesics from x:



Under $(*)$: $\xi^*(x)$ is the asymptotic direction of the competition interface between subtrees rooted at $x+e_1$ and $x+e_2$

$\operatorname{supp}\mu_{x,y}$ = isolated points = jumps of $\xi \to B^{\xi}(x,y)$

$\iff \gamma^{x,\xi-}$ coalesce and $\gamma^{x,\xi+}$ coalesce

$\operatorname{supp}\mu_{x,y}$ = isolated points = jumps of $\xi \to B^{\xi}(x,y)$

$\iff \gamma^{x,\xi-}$ coalesce and $\gamma^{x,\xi+}$ coalesce

$\operatorname{supp}\mu_{x,y}$ = isolated points = jumps of $\xi \to B^{\xi}(x,y)$

$\iff \gamma^{x,\xi-}$ coalesce and $\gamma^{x,\xi+}$ coalesce

Then $\mathcal{V}^{\omega} = \{\xi^*(x) : x \in \mathbb{Z}^2\}$! (Countable)

$\operatorname{supp}\mu_{x,y}$ = isolated points = jumps of $\xi \to B^{\xi}(x,y)$

$\Longleftrightarrow$ $\gamma^{x,\xi-}$ coalesce and $\gamma^{x,\xi+}$ coalesce

Then $\mathcal{V}^{\omega} = \{\xi^*(x) : x \in \mathbb{Z}^2\}$! (Countable)

<u>Special case:</u> $\omega_x \sim \exp(1)$

$\mathrm{supp}\,\mu_{x,y}$ = isolated points = jumps of $\xi \to B^{\xi}(x,y)$

$\iff$ $\gamma^{x,\xi-}$ coalesce and $\gamma^{x,\xi+}$ coalesce

Then $\mathcal{V}^{\omega} = \{\xi^*(x) : x \in \mathbb{Z}^2\}$! (Countable)

<u>Special case:</u> $\omega_x \sim \exp(1)$

$g(\xi) = (\sqrt{\xi_1} + \sqrt{\xi_2})^2$ implies (*) holds

$\operatorname{supp}\mu_{x,y}$ = isolated points = jumps of $\xi \to B^{\xi}(x,y)$

$\iff \gamma^{x,\xi-}$ coalesce and $\gamma^{x,\xi+}$ coalesce

Then $\mathcal{V}^{\omega} = \{\xi^*(x) : x \in \mathbb{Z}^2\}$! (Countable)

<u>Special case:</u> $\omega_x \sim \exp(1)$

$g(\xi) = (\sqrt{\xi_1} + \sqrt{\xi_2})^2$ implies (*) holds

Fan & Seppäläinen '19: $\operatorname{supp}\mu_{x,y}$ = union of Poisson processes

$\operatorname{supp}\mu_{x,y}$ = isolated points = jumps of $\xi \to B^{\xi}(x,y)$

$\iff \gamma^{x,\xi-}$ coalesce and $\gamma^{x,\xi+}$ coalesce

Then $\mathcal{V}^{\omega} = \{\xi^*(x) : x \in \mathbb{Z}^2\}$! (Countable)

<u>Special case:</u> $\omega_x \sim \exp(1)$

$g(\xi) = (\sqrt{\xi_1} + \sqrt{\xi_2})^2$ implies $(*)$ holds

Fan & Seppäläinen '19: $\operatorname{supp}\mu_{x,y}$ = union of Poisson processes

Coupier '11: $\nexists\, \xi$ s.t.

Theorem: $\omega_x \sim \exp(1)$. The following hold a.s.

Theorem: $\omega_x \sim \exp(1)$. The following hold a.s.

- $\mathcal{V}^{\omega} = \{\xi^*(x) : x \in \mathbb{Z}^2\}$ = jumps of $B^{\bullet}(x,y)$, $x,y \in \mathbb{Z}^2$

  countable+dense

<u>Theorem:</u> $\omega_x \sim \exp(1)$. The following hold a.s.

- $\mathcal{V}^\omega = \{\xi^*(x) : x \in \mathbb{Z}^2\}$ = jumps of $B^\bullet(x,y)$, $x,y \in \mathbb{Z}^2$
  countable+dense

- $\xi \notin \mathcal{V}^\omega$: $\forall x$ $\exists!$ ξ-dir. geo. out of x. They all coalesce

<u>Theorem:</u> $\omega_x \sim \exp(1)$. The following hold a.s.

- $\mathcal{V}^\omega = \{\xi^*(x) : x \in \mathbb{Z}^2\}$ = jumps of $B^\cdot(x,y)$, $x,y \in \mathbb{Z}^2$
  countable+dense
- $\xi \notin \mathcal{V}^\omega$: $\forall x$ $\exists!$ ξ-dir. geo. out of x. They all coalesce
- $\xi \in \mathcal{V}^\omega$: $\forall x$ $\exists$ <u>exactly</u> two ξ-dir. geo. out of x: $\gamma^{x,\xi\pm}$

  $\gamma^{\cdot,\xi-}$ all coalesce & $\gamma^{\cdot,\xi+}$ all coalesce

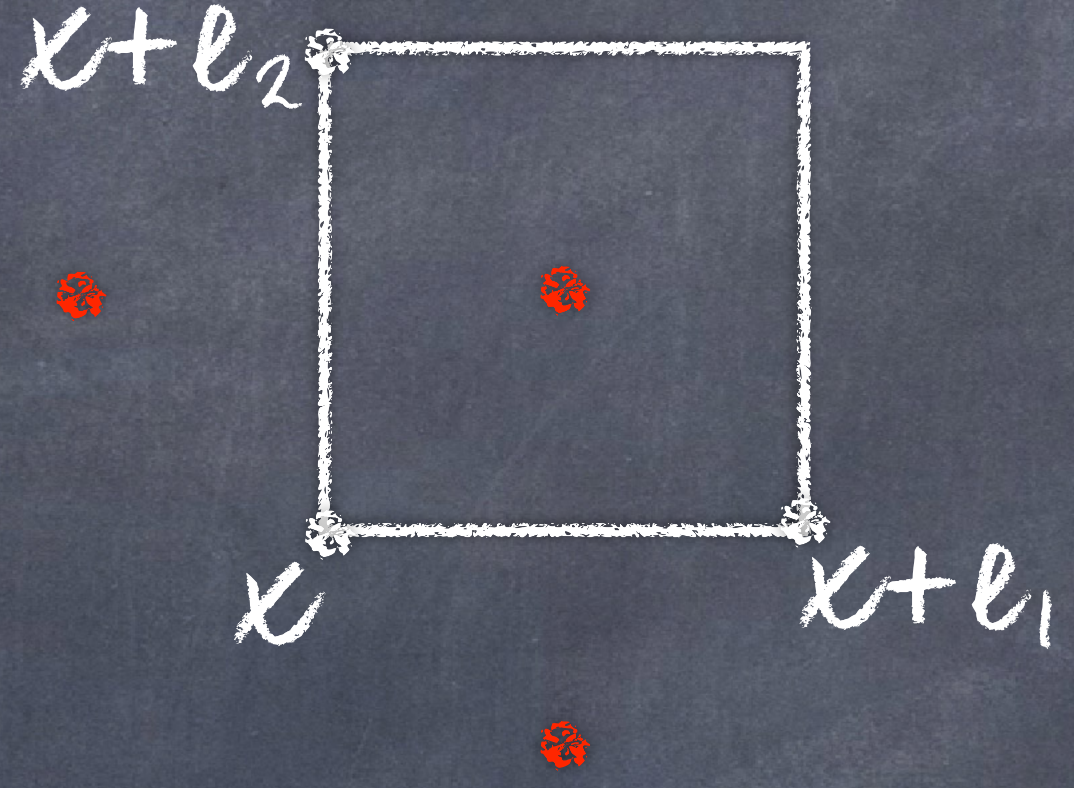
<u>Theorem:</u> $\omega_x \sim \exp(1)$. The following hold a.s.

- $\mathcal{V}^\omega = \{\xi^*(x) : x \in \mathbb{Z}^2\}$ = jumps of $B^\bullet(x,y)$, $x,y \in \mathbb{Z}^2$
  countable+dense
- $\xi \notin \mathcal{V}^\omega$: $\forall x$ $\exists!$ ξ-dir. geo. out of x. They all coalesce
- $\xi \in \mathcal{V}^\omega$: $\forall x$ $\exists$ <u>exactly</u> two ξ-dir. geo. out of x: $\gamma^{x,\xi\pm}$

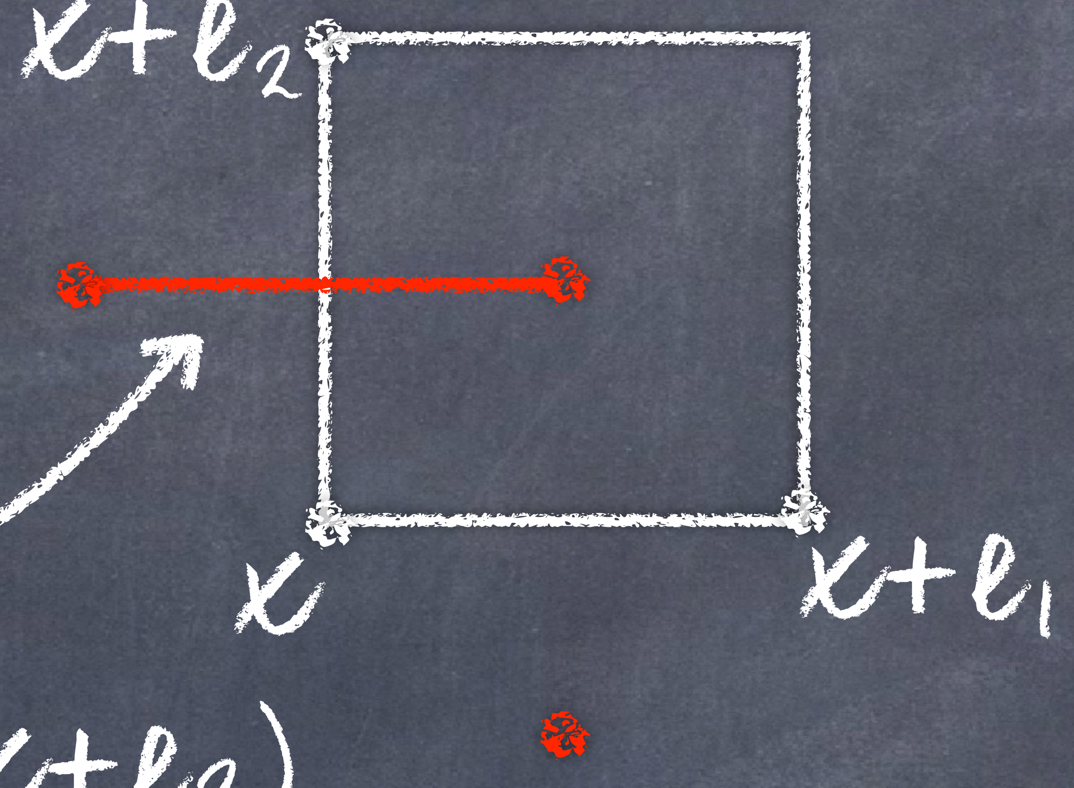
  $\gamma^{\cdot,\xi-}$ all coalesce & $\gamma^{\cdot,\xi+}$ all coalesce
- That's it! No other semi-infinite geodesics

Fix $\xi \in V^{\omega}$ and consider the following graph on the dual lattice:

Fix $\xi \in V^{\omega}$ and consider the following graph on the dual lattice:



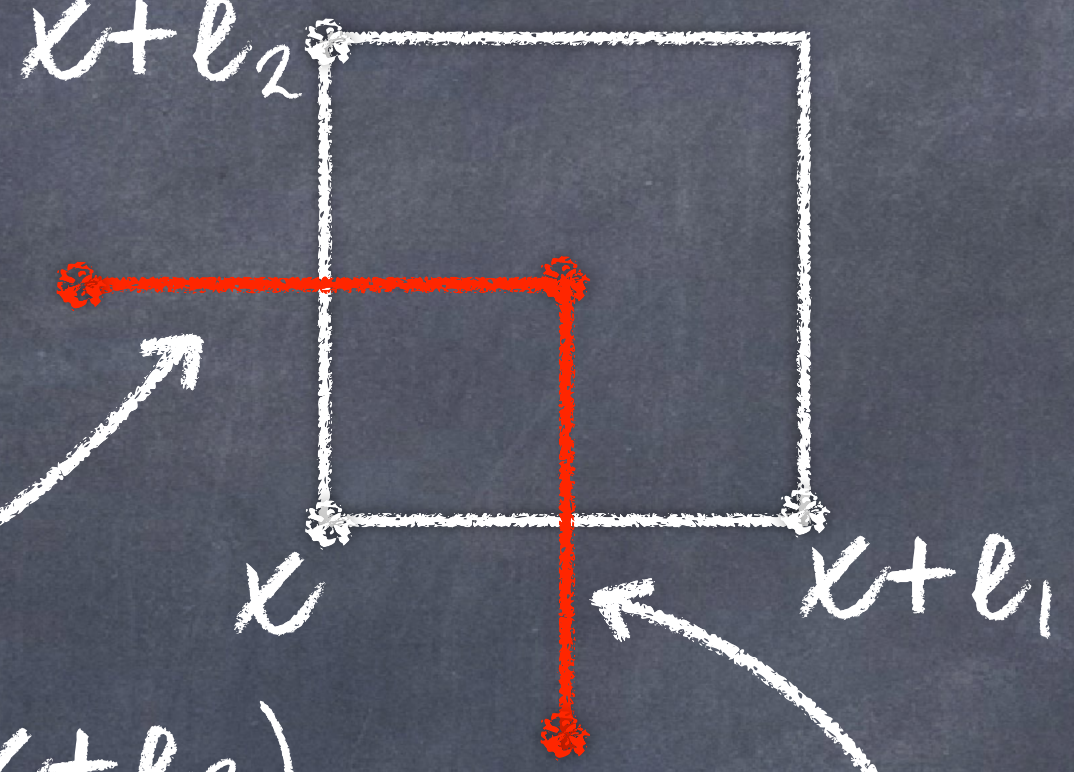
Fix $\xi \in V^\omega$ and consider the following graph on the dual lattice:



present if

$B^{\xi+}(x,x+e_2) \neq B^{\xi-}(x,x+e_2)$

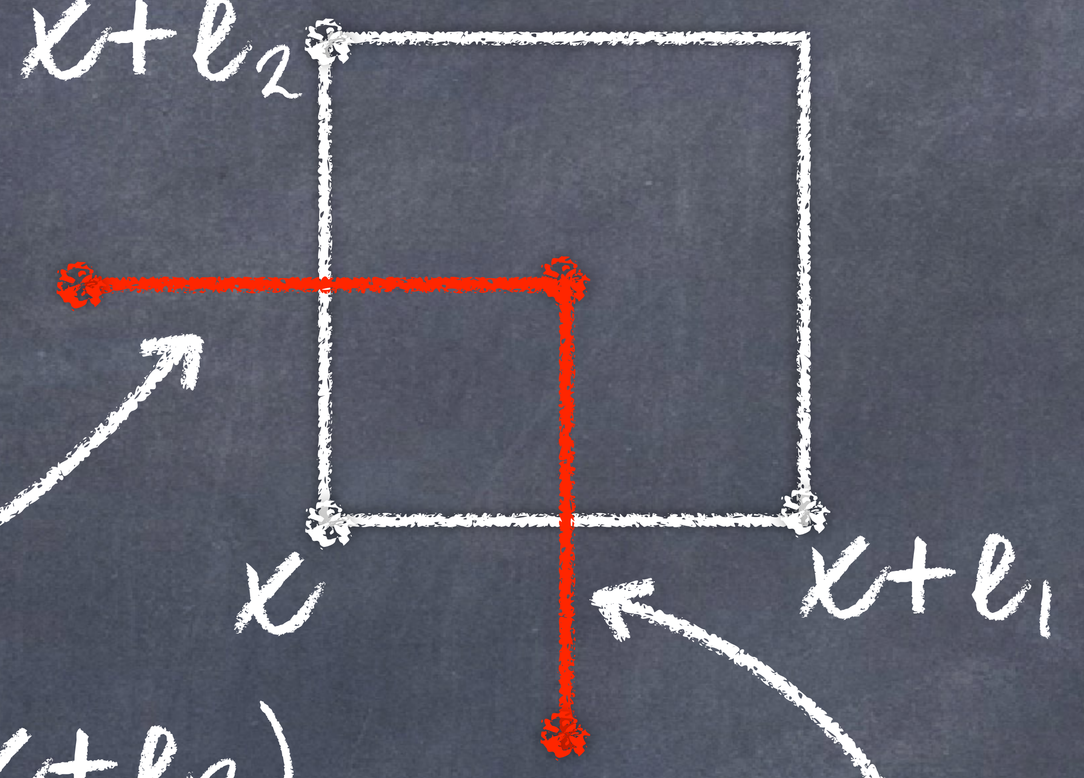
Fix $\xi \in V^{\omega}$ and consider the following graph on the dual lattice:



present if
$B^{\xi+}(x,x+e_2) \neq B^{\xi-}(x,x+e_2)$

present if
$B^{\xi+}(x,x+e_1) \neq B^{\xi-}(x,x+e_1)$

Fix $\xi \in \mathcal{V}^{\omega}$ and consider the following graph on the dual lattice:

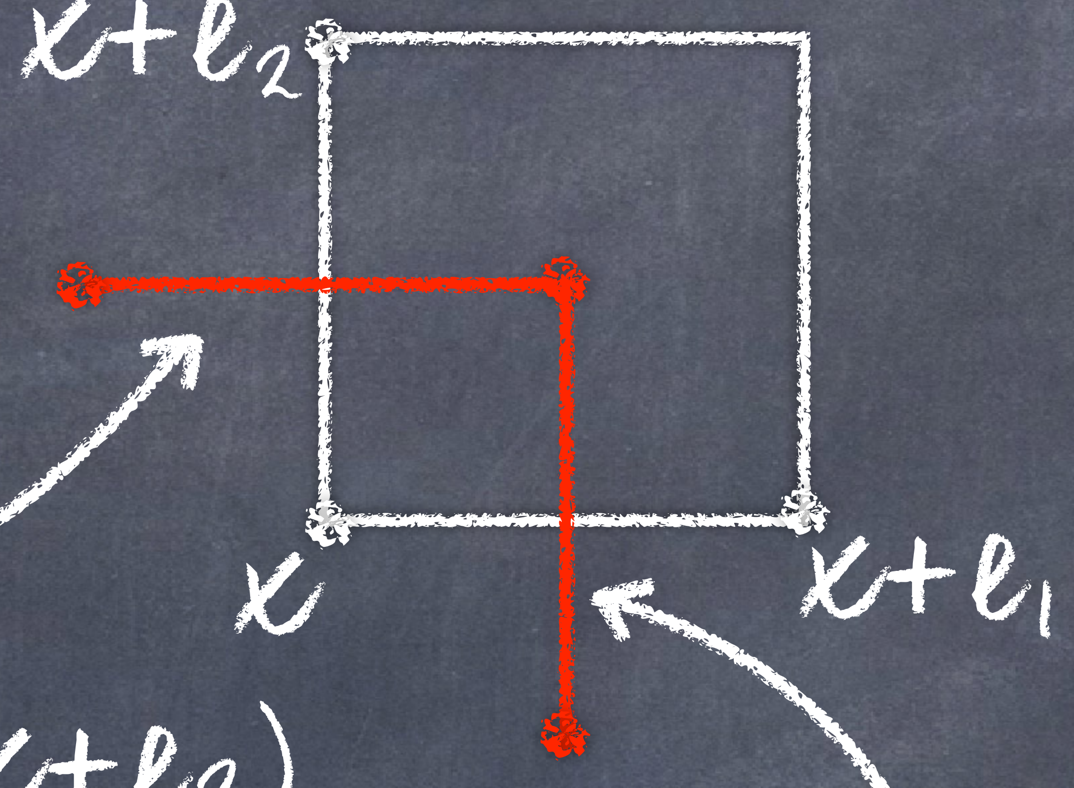


present if
$B^{\xi+}(x,x+e_2) \neq B^{\xi-}(x,x+e_2)$

present if
$B^{\xi+}(x,x+e_1) \neq B^{\xi-}(x,x+e_1)$

<u>Properties:</u> bi-infinite paths, coalescing both ↙ & ↗

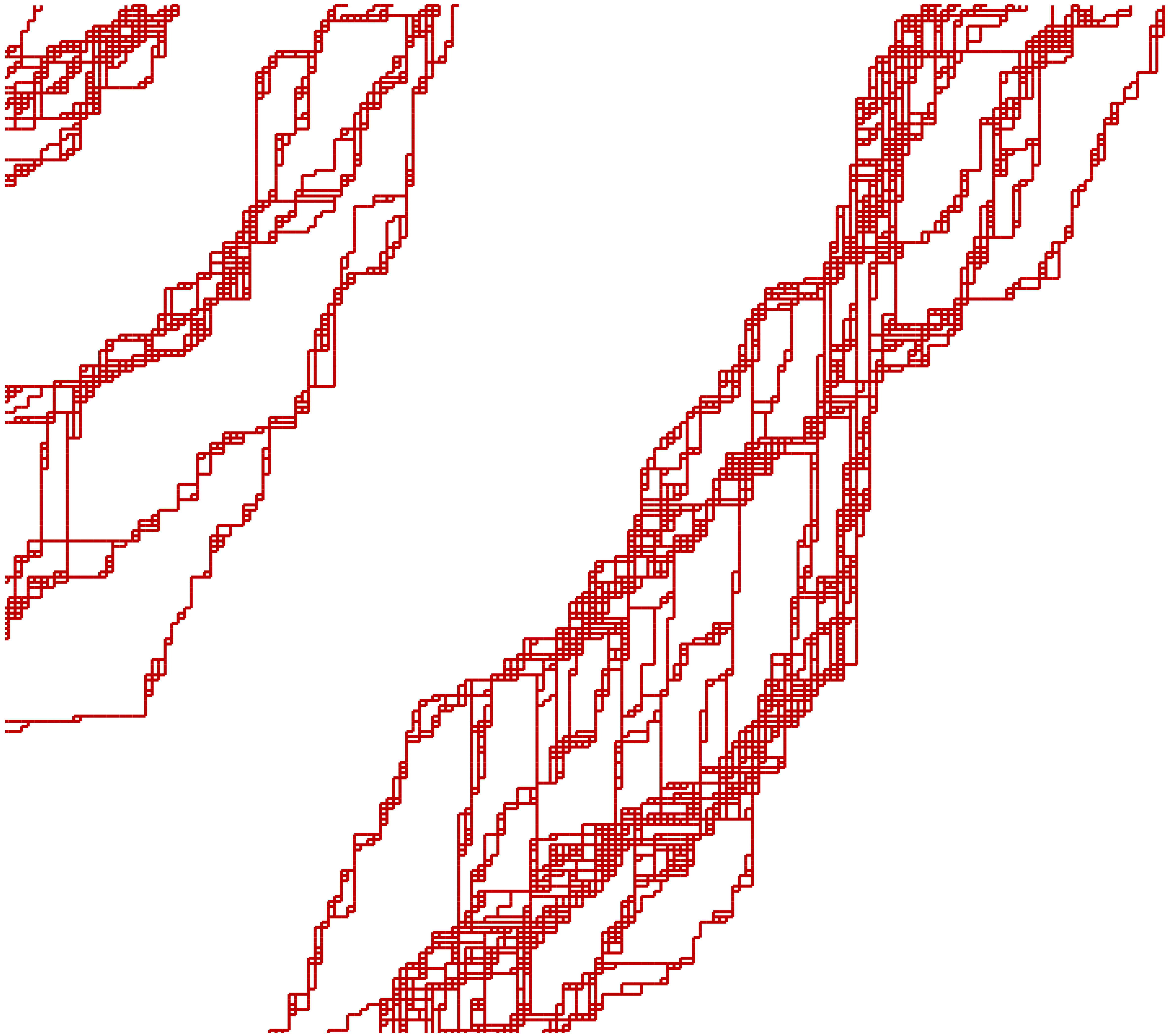
Fix $\xi \in \mathcal{V}^{\omega}$ and consider the following graph on the dual lattice:

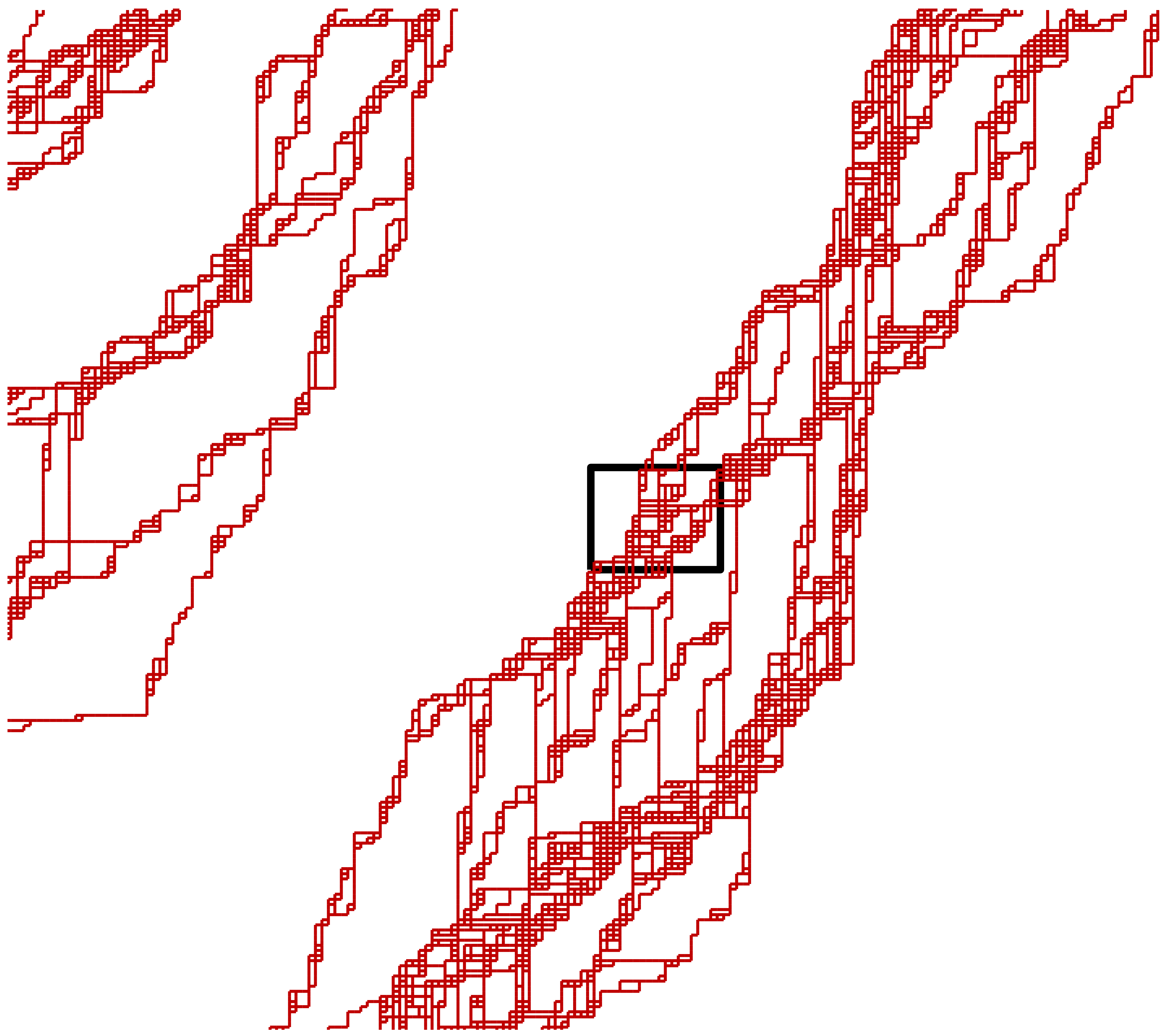


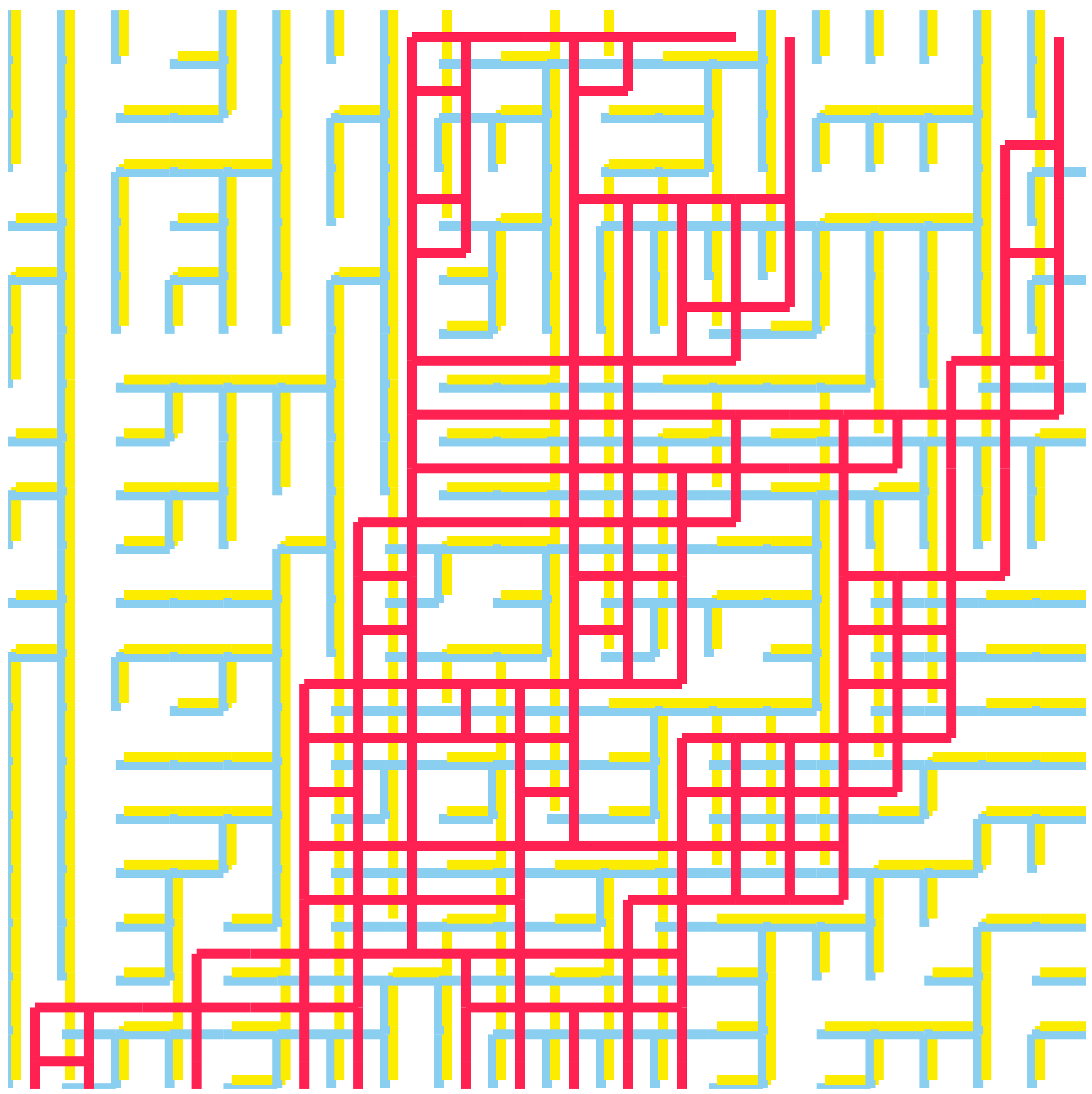
present if
$B^{\xi+}(x,x+e_2) \neq B^{\xi-}(x,x+e_2)$

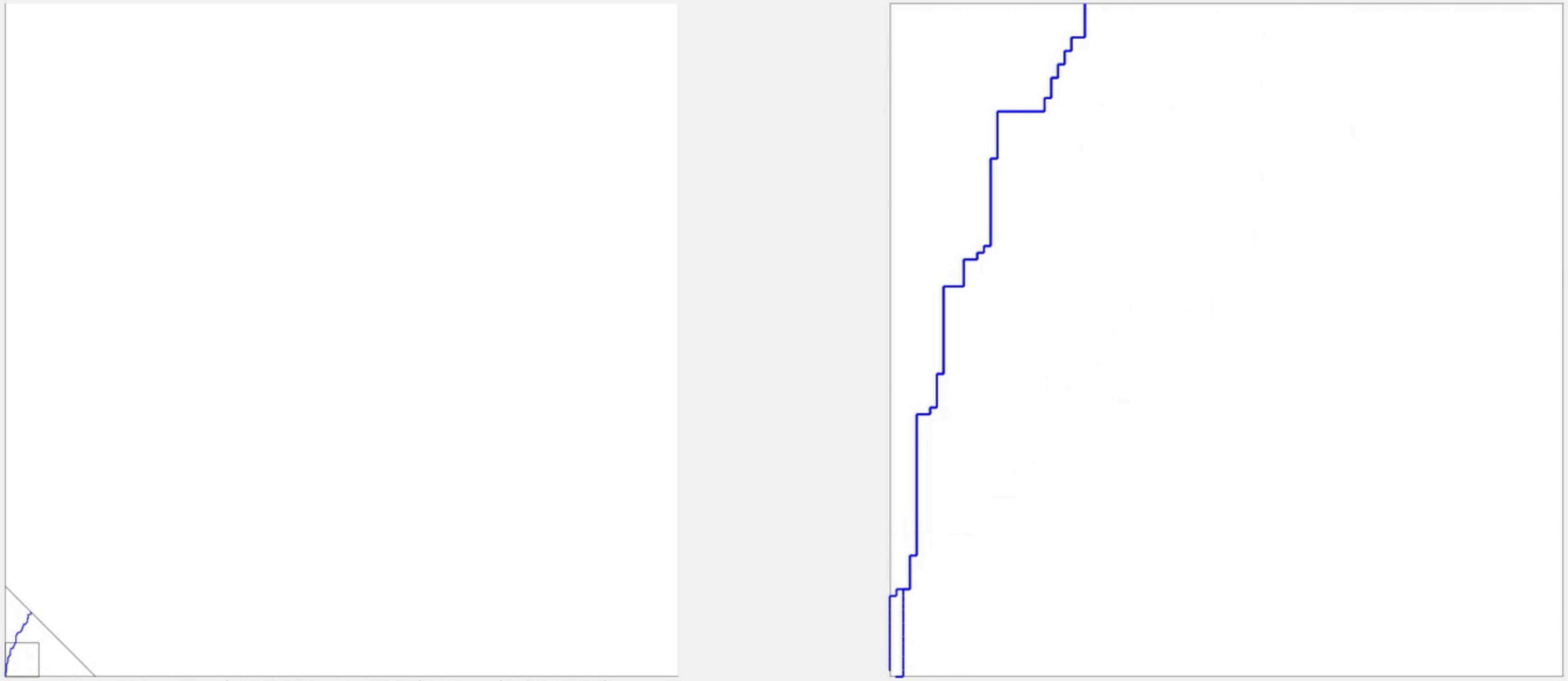
present if
$B^{\xi+}(x,x+e_1) \neq B^{\xi-}(x,x+e_1)$

<u>Properties:</u> bi-infinite paths, coalescing both ↙ & ↗
has 0 density, union of dual trees of $\xi-$ & $\xi+$ trees



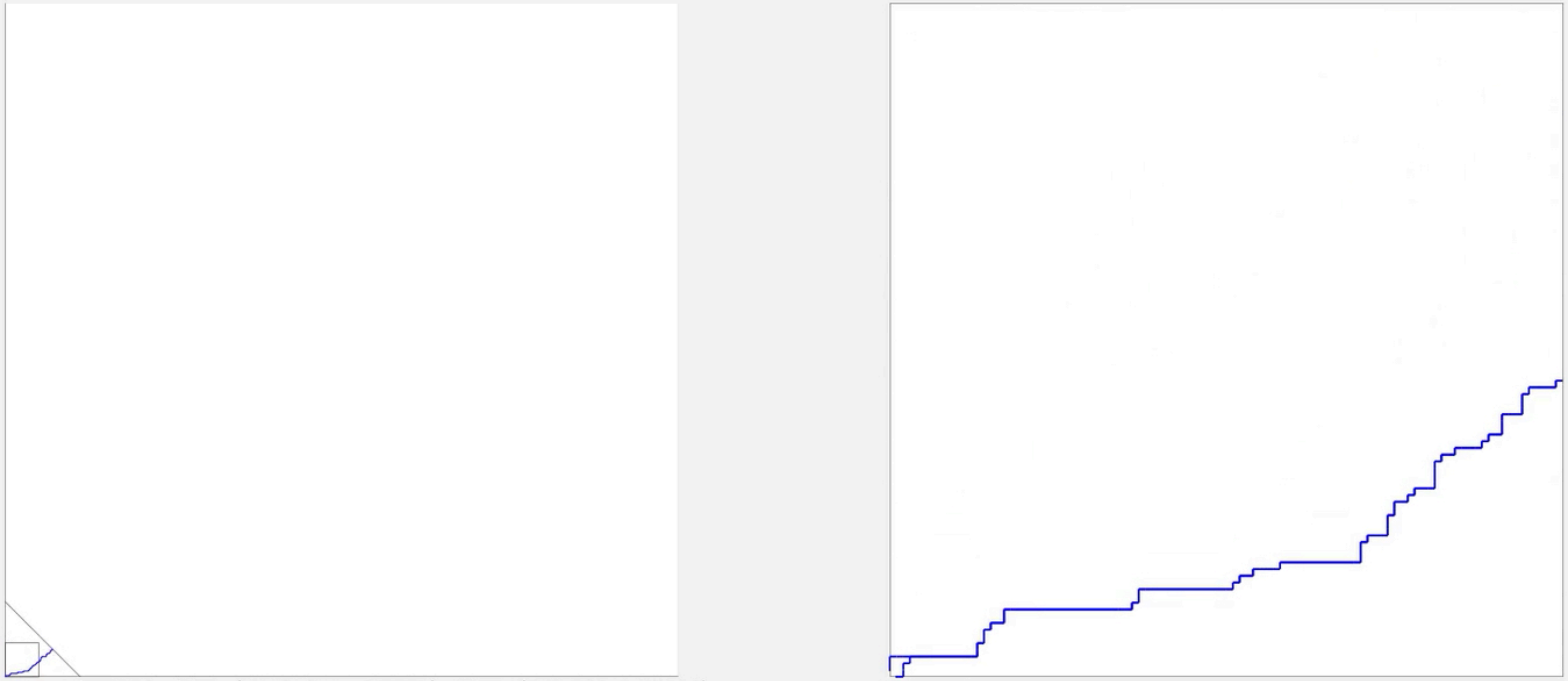






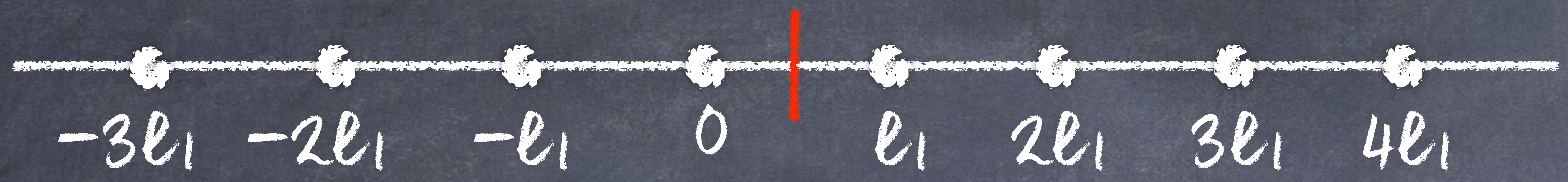
$\ell = 269,\ h = (-2.3628, -1.7338),\ \xi = (0.35, 0.65)$

$\ell = 1,\ h = (-2.3628, -1.7338),\ \xi = (0.35, 0.65)$

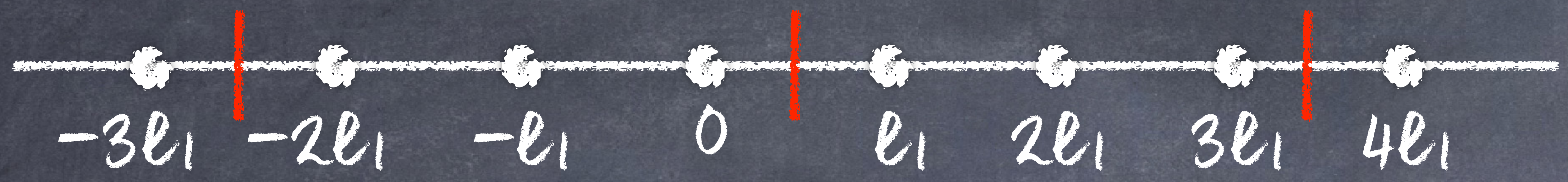


$\ell = 223,\ h^* = (-1.9174, -2.0901),\ \xi^* = (0.54302, 0.45698)$


−3ℓ1
−2ℓ1
−ℓ1
0
ℓ1
2ℓ1
3ℓ1
4ℓ1

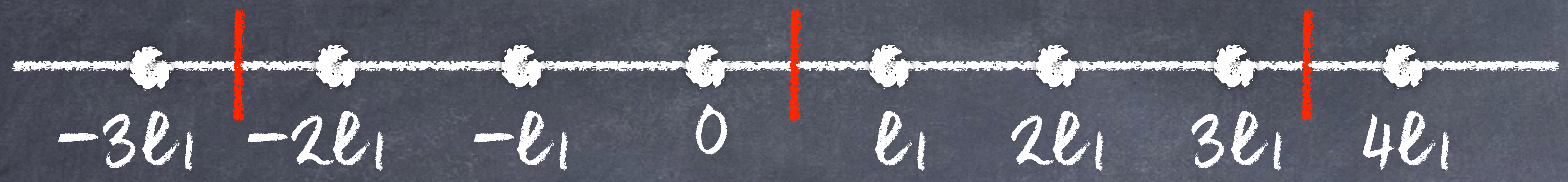


Conditional (Palm sense) on $B^{\xi+}(0,\ell_1) \neq B^{\xi-}(0,\ell_1)$



Conditional (Palm sense) on $B^{\xi+}(0,\ell_1)\neq B^{\xi-}(0,\ell_1)$

the locations of the other jumps/crossings



Conditional (Palm sense) on $B^{\xi+}(0,e_1) \neq B^{\xi-}(0,e_1)$

the locations of the other jumps/crossings are $\{k: S_{2k}=0\}$ where S_n is a SSRW !