Absorbing-state phase transitions

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Background

Context

Non-equilibrium Statistical Mechanics

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Non-equilibrium Statistical Mechanics

Critical phenomena

self-similar shapes

large fluctuations

long-range correlations

avalanches

Critical phenomena

Many physical systems behave as:

$\beta < \beta_c \;\rightarrow$ well behaved, short-range correlations

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$\beta < \beta_c \ \rightarrow$ well behaved, short-range correlations

$\beta = \beta_c \ \rightarrow$ long range, power laws, intricate behavior

$\beta > \beta_c \ \rightarrow$ well behaved, short-range correlations

But why do we observe critical behavior outside a controlled environment?

Critical phenomena (cont)

Physics literature:

Late 80's – Self-organized criticality

A system that finds a critical state all by itself

Critical phenomena (cont)

Physics literature:

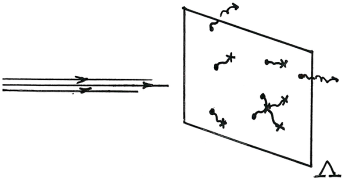
Late 80's – Self-organized criticality

A system that finds a critical state all by itself

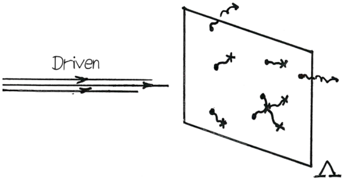
Late 90's – Relate it to ordinary phase transitions

Absorbing-state phase transitions

Driven-dissipative dynamics



Driven-dissipative dynamics



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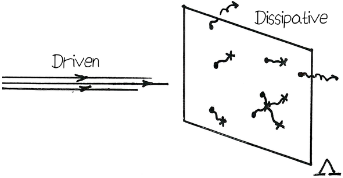

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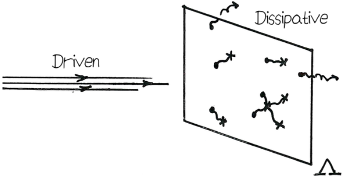

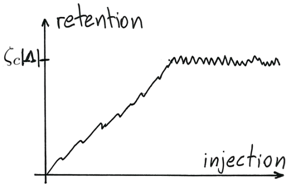

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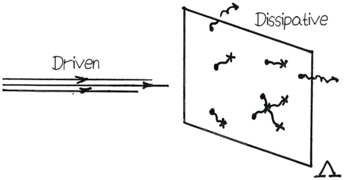

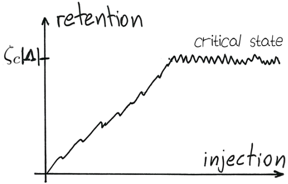

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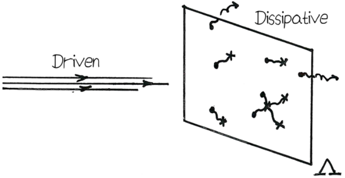
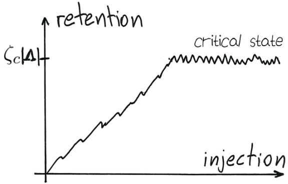
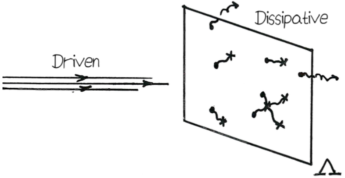
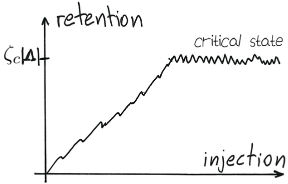




Infinite-volume conservative system

Driven-dissipative dynamics



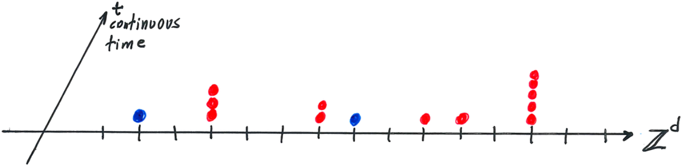


Infinite-volume conservative system

System goes to an absorbing state if the density of particles is below ζ_c, and remains unstable if the density is above ζ_c

Model and predictions

Activated Random Walks



	rate	
A → S	λ	
A jumps	1	from x to $x+y$ with y chosen randomly
AS → AA	∞	

Assumptions

- Jumps to nearest-neighbors

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- Particles start active
- At $t = 0$, i.i.d. Poisson(ζ) particles
- $0 < \lambda \leqslant \infty$

Fixation vs activity

Fixation: each site is eventually stable

Activity: each site is visited infinitely many times

Fixation vs activity

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Dichotomy: either fixation a.s. or activity a.s.

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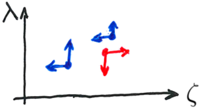
Monotonicity:



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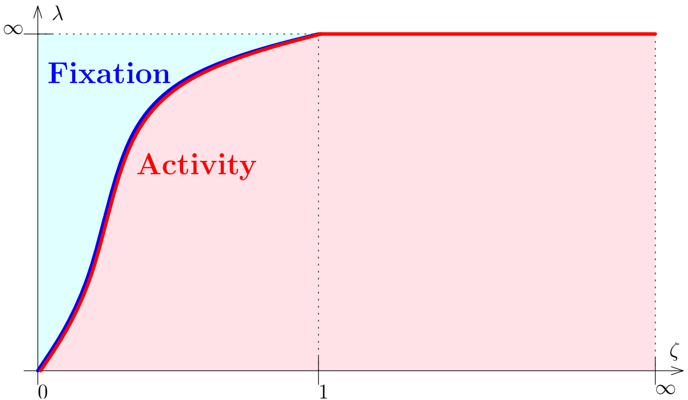

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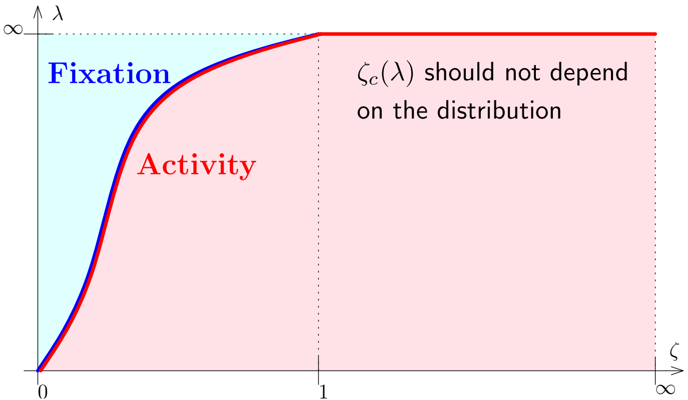

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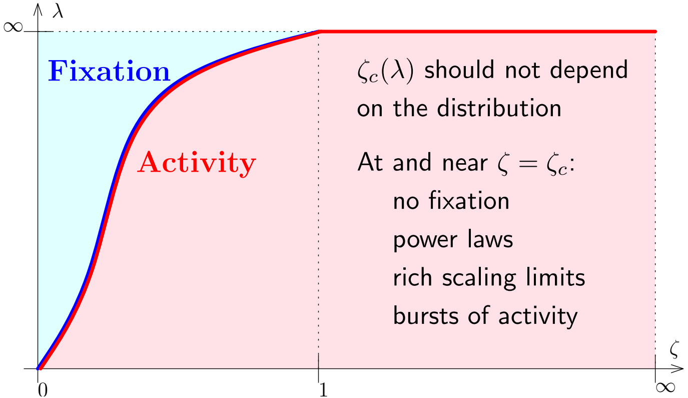

Difficulties

Lack of attractiveness

Overcome by using constructions other than Harris'

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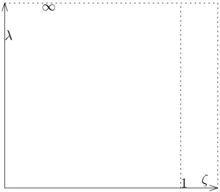
Lack of attractiveness

Overcome by using constructions other than Harris'

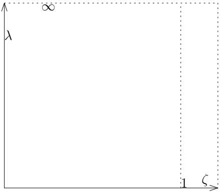
Conservation of particles

Rules out "energy vs. entropy" approaches

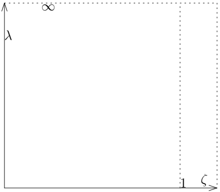
Phase transition results



$d = 1$ directed



$d = 1$ biased



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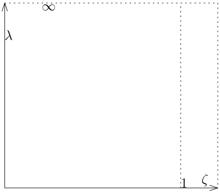
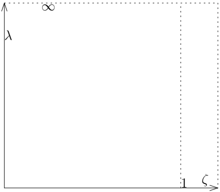
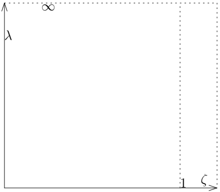


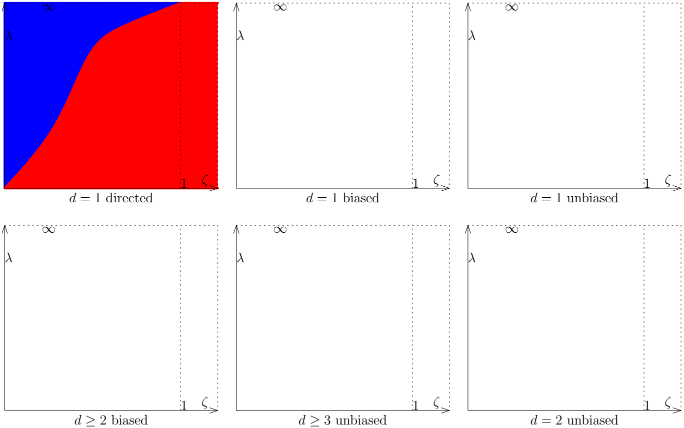
$d \geq 2$ biased



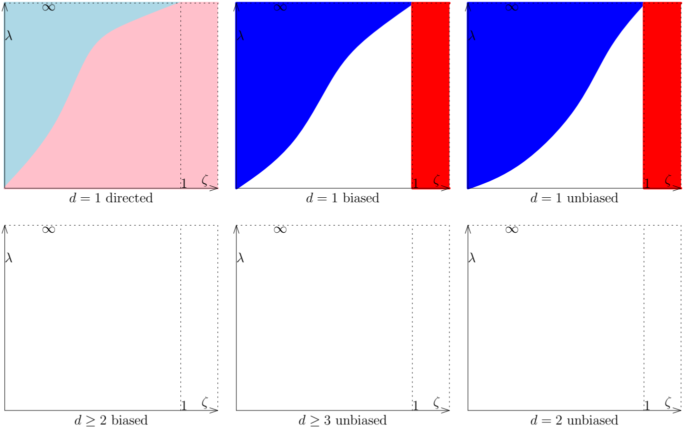
$d \geq 3$ unbiased



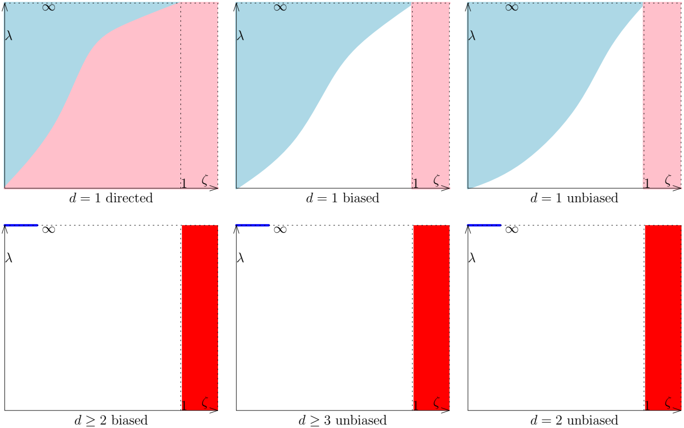
$d = 2$ unbiased



Hoffman, Sidoravicius. *Unpublished* (2004) | Cabezas, R, Sidoravicius. **J Stat Phys** (2014)



R, Sidoravicius. **Invent Math** (2012)



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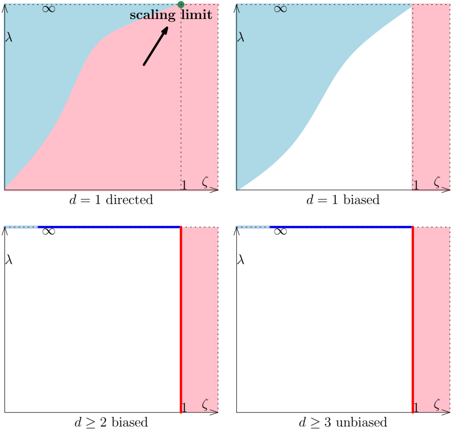

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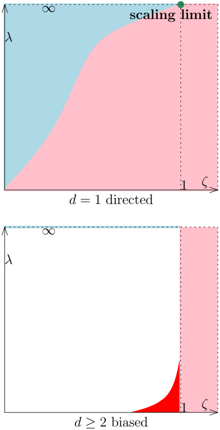
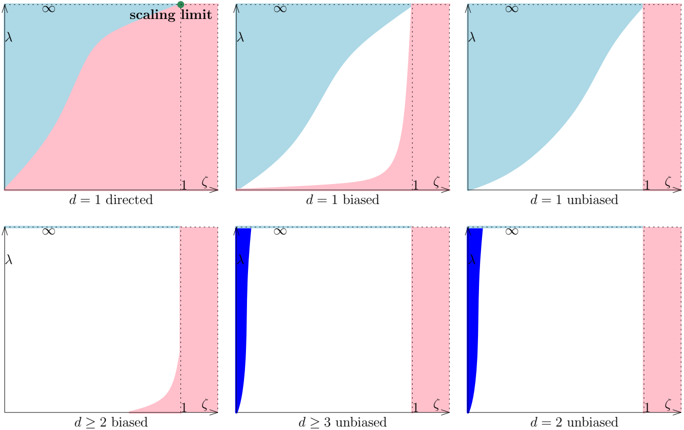


Taggi. **Electron J Probab** (2016)



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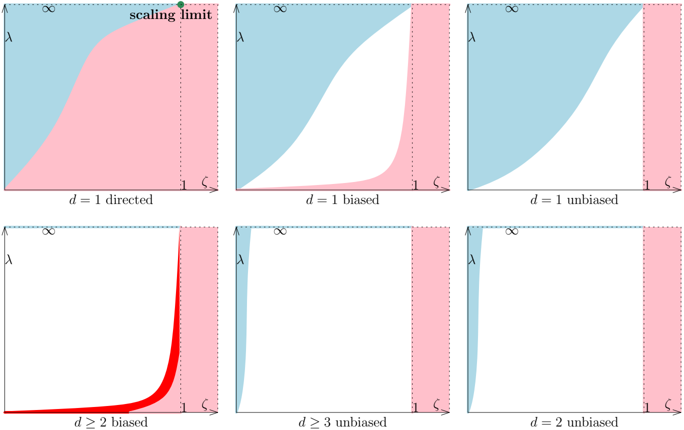

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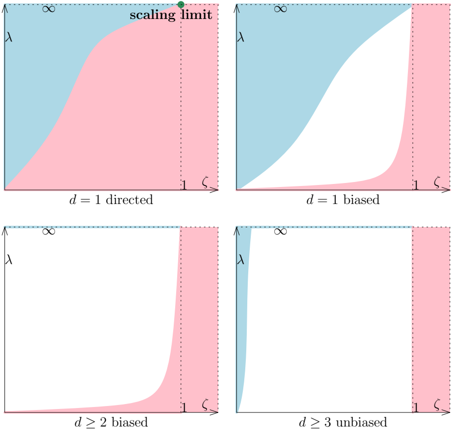

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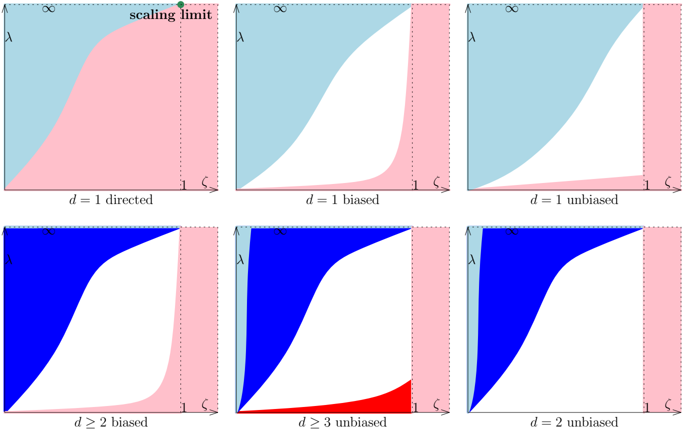

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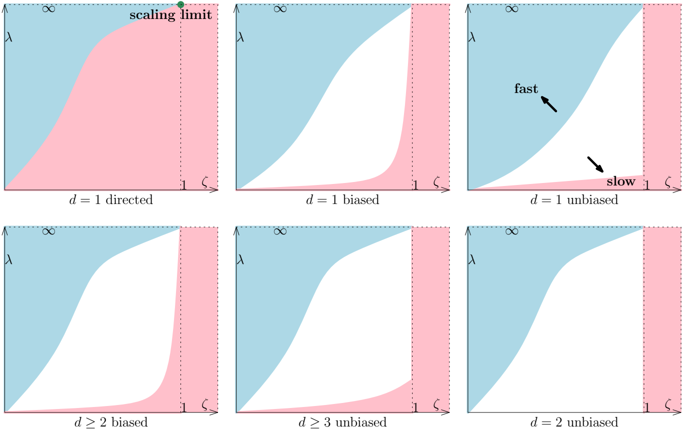

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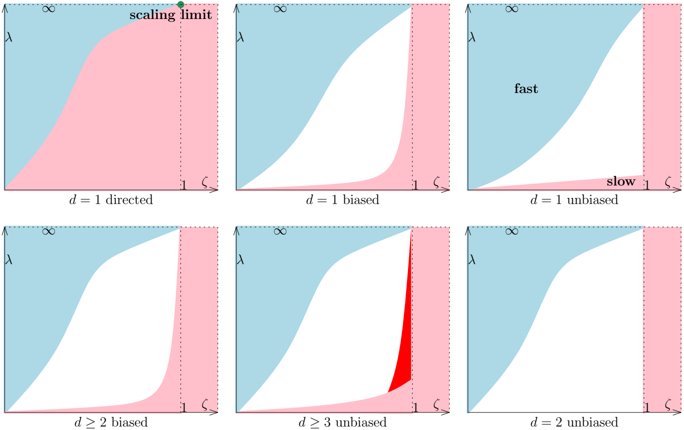

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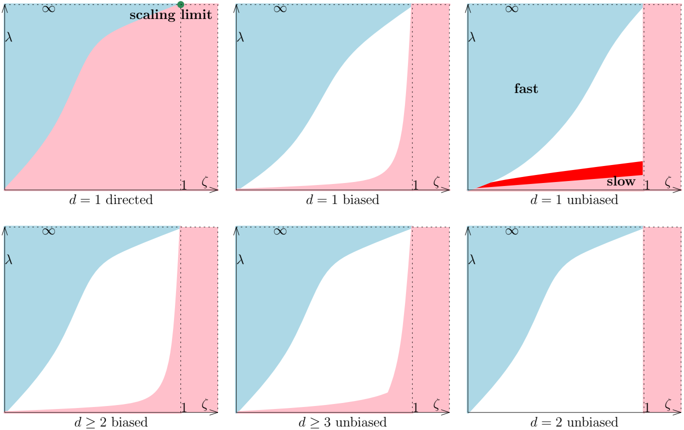

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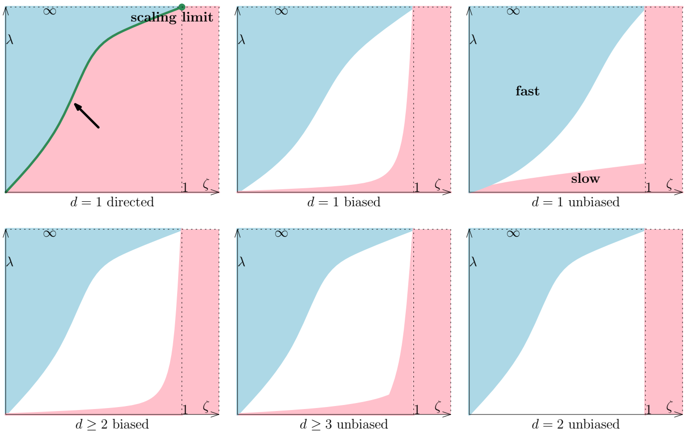

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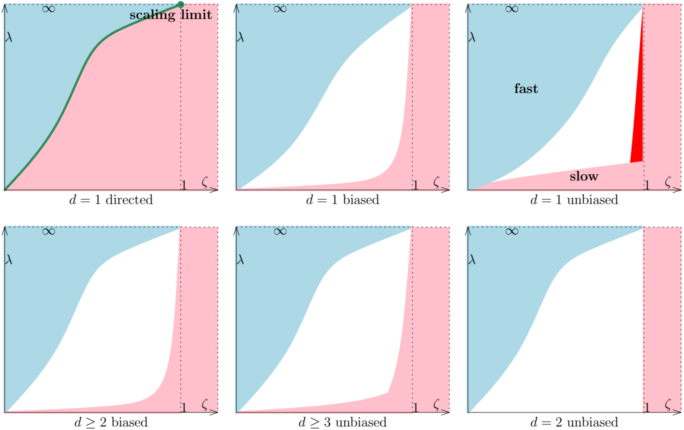
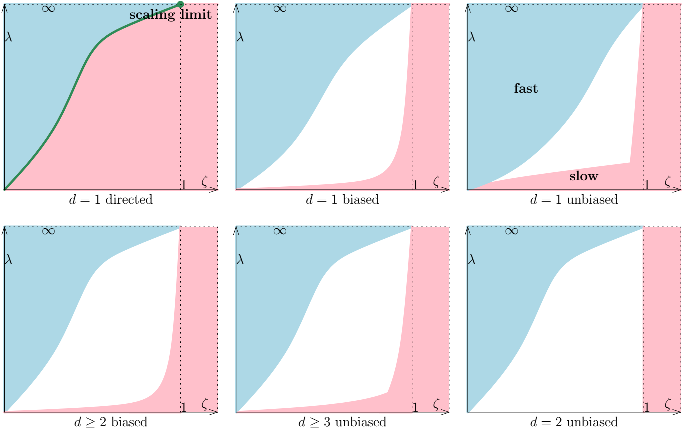


Hoffman, Richey, R. *Writing up*



∞
λ
scaling limit
1
ζ
$d = 1$ directed
$d = 1$ biased
fast
slow
$d = 1$ unbiased
$d \geq 2$ biased
$d \geq 3$ unbiased
$d = 2$ unbiased

Constructions and main tools

Constructions

Harris graphical construction

clocks with marks at each site

Constructions

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clocks with marks at each site

Site-wise construction

stack of instructions at each site

Constructions

Harris graphical construction

clocks with marks at each site

Site-wise construction

stack of instructions at each site

Particle-wise constructions

particles start with a life plan and do pause/resume

Main tools

Site-wise representation and Abelianess

Diaconis, Fulton. **Rend Semin Mat Torino** (1991) | Eriksson. **SIAM J Discrete Math** (1996)

Main tools

Site-wise representation and Abelianess

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Relate it to the dynamics, preserving monotonicity, etc

Reduce fixation-activity question to *toppling procedures*

R, Sidoravicius. **Invent Math** (2012)

Main tools (cont)

Assuming a particle-wise construction is well-defined:

a particle stays active $\Rightarrow$ sites stay active

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Well-definedness of the particle-wise construction

$\rightarrow$ ergodicity, mass transport, coupling, surgery

Averaged criterion for activity

R, Tournier. **Ann Inst H Poincaré Probab Statist** (2018)

Criterion for activity

$\vdash$ Stabilizing a large box forces a large number of particles to **visit a specific site, wpp**

R, Sidoravicius. **Invent Math** (2012)

Criterion for activity

$\vdash$ Stabilizing a large box forces a large number of particles to **visit a specific site, wpp**

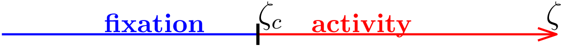
R, Sidoravicius. **Invent Math** (2012)

$\vdash$ Stabilizing a large box forces a positive fraction of the particles to **leave the box, on average**

R, Tournier. **Ann Inst H Poincaré Probab Statist** (2018)

Sharpness

Theorem. Given d, λ, p, there exists ζ_c such that



for all ergodic initial states with density ζ

R, Sidoravicius, Zindy. **Ann Henri Poincaré** (2019)

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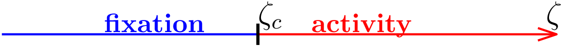

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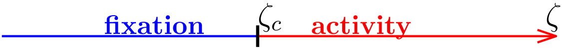

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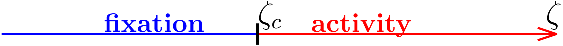

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– Can drop the previous "i.i.d. Poisson" assumption

– Restrictive proofs now yield general theorems

– Contributes to ongoing discussion about some dissipative models mixing better than others

Scaling limits and avalanches

Critical one-dimensional model

$d = 1$, directed walks (integrable case)

i.i.d. initial condition with critical density $\zeta = \zeta_c = \frac{\lambda}{1+\lambda}$

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Run the dynamics on $V_n = [0, n]$ until it is stable.
$C_n :=$ how many particles cross the origin.

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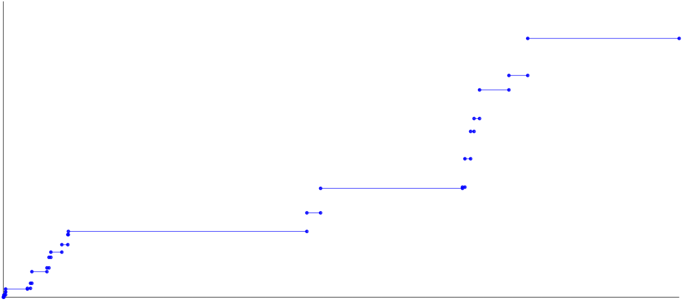
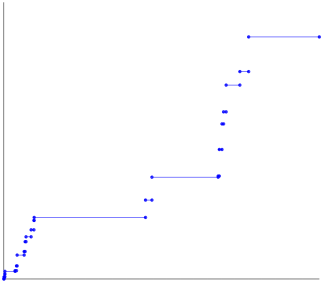
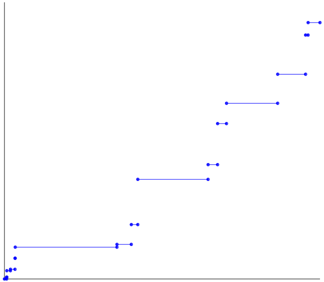
Run the dynamics on $V_n = [0, n]$ until it is stable.
$C_n :=$ how many particles cross the origin.

Released the active particles at $x = n + 1$, let them interact with the leftovers of previous step. $C_{n+1} \geq C_n$

Simulation







Scaling limit

Theorem (Cabezas, R '19).
The counting process $(C_n)_{n\geqslant 0}$ rescales to $(\mathcal{C}^{\rho}_x)_{x\geqslant 0}$

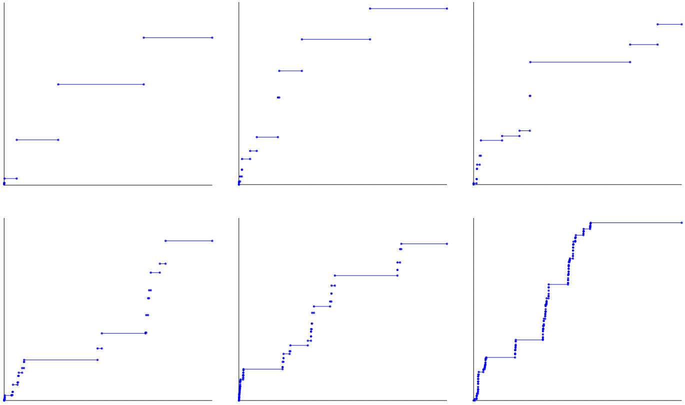
Pure-jump process constructed from a collection of correlated reflected coalescing Brownian motions

Scaling limit

Theorem (Cabezas, R '19).
The counting process $(C_n)_{n \geqslant 0}$ rescales to $(\mathcal{C}^{\rho}_x)_{x \geqslant 0}$

Pure-jump process constructed from a collection of correlated reflected coalescing Brownian motions

Correlations depend on $\rho = \frac{\sigma_s}{\sigma_p} \in (0, 1]$, where $\sigma_s^2 = \zeta - \zeta^2$ and σ_p^2 is the variance of initial condition



$\rho = 1.00,\ 0.50,\ 0.30,\ 0.10,\ 0.05,\ 0.00$

Notes on some selected proofs

Toppling procedures

Abelian particle-wise construction

$m_V(x) :=$ odometer at x when V is stabilized

$$\lim_k \lim_V \mathbb{P}[m_V(x) \geqslant k] = \begin{cases} 0 \Leftrightarrow \text{Fixation} & \text{(B)} \\ 1 \Leftrightarrow \text{Activity} & \text{(U)} \end{cases}$$

$M_n := \#$Particles which quit when B_n is stabilized

$$\limsup_n \frac{\mathbb{E}[M_n]}{|B_n|} > 0 \;\Rightarrow\; \text{Activity} \quad \text{(E)}$$

Examples

Thm (Stauffer, Taggi)**.** $\zeta_c \geqslant \frac{\lambda}{1+\lambda}$

Thm (Taggi; R, Tournier)**.** $\zeta_c \leqslant F_p(\lambda)$

Particle-wise construction

Labeled particles $\to$ mass transport, ergodicity, surgery

Construction: assign to each particle a CTRW+beep

MTP Example. Assume particles fixate a.s.

$$\zeta = \mathbb{E}[\text{start at } \mathbf{o}] = \mathbb{E}[\text{settle at } \mathbf{o}] \leqslant 1$$

Thm. Site fixation $\Rightarrow \zeta < 1$

Cabezas, R, Sidoravicius. **Probab Theory Relat Fields** (2018)

Particle-wise construction (cont)

Thm. The PWC is well-defined

Add particles one by one, updating the whole evolution

$\vdash$ Life of each particle is well-defined through some limit

Main step: $\forall x, T$, the number of particle additions that affect site x by time T has finite expectation

R, Tournier. **Ann Inst H Poincaré Probab Statist** (2018)

Open problems and questions

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Proofs of fixation/activity that give different insights

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Unbiased walks on $\mathbb{Z}^2$: $\zeta_c < 1$ for some $\lambda < \infty$

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Unbiased walks on $\mathbb{Z}^2$: $\zeta_c < 1$ for some $\lambda < \infty$

Dichotomy for the slow-fast transition on finite ring

Open problems and questions

Proofs of fixation/activity that give different insights

Unbiased walks on $\mathbb{Z}^2$: $\zeta_c < 1$ for some $\lambda < \infty$

Dichotomy for the slow-fast transition on finite ring

Make sense of scaling limit at criticality for $d \geqslant 2$

Open problems and questions

Proofs of fixation/activity that give different insights

Unbiased walks on $\mathbb{Z}^2$: $\zeta_c < 1$ for some $\lambda < \infty$

Dichotomy for the slow-fast transition on finite ring

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Sharpness when a fraction of particles start active

Open problems and questions

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Many more...