Random matrices, operators and analytic functions

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joint with B. Virág (Toronto)

Eugene Wigner, 1950s:

- *energy levels of heavy nuclei*

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  - *the spectrum of a complicated self-adjoint operator*

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In this talk:

random matrices $\rightsquigarrow$ (random) operators

Basic question of RMT:

What can we say about the spectrum of a large random matrix?

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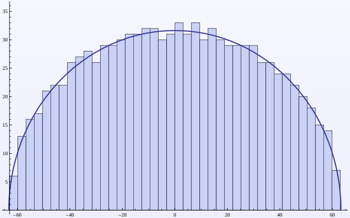

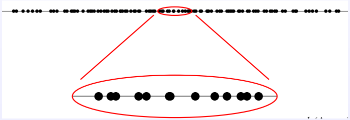

global

local

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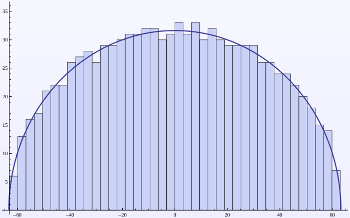

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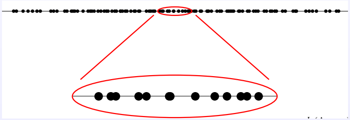

local

How about other observables related to the matrix?

A classical example: Gaussian Unitary Ensemble

$M = \frac{A+A^*}{\sqrt{2}}$, $\quad$ A is $n \times n$ with iid standard complex normal entries

$A_{i,j}$ has density $\frac{1}{\pi} e^{-|x|^2}$ on $\mathbb{C}$

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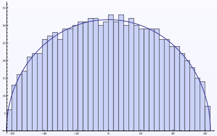

Global picture: Wigner semicircle law

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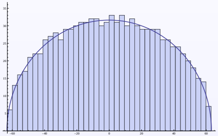
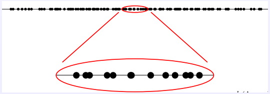


Global picture: Wigner semicircle law

Local picture: point process limit in the bulk and near the edge



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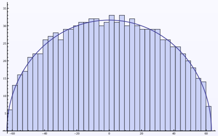

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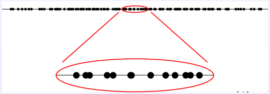

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The limit processes are characterized by their joint intensity functions.
Roughly: how likely that we see random points near certain values.

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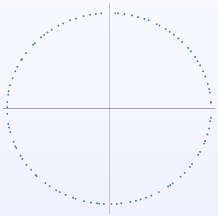
Eigenvalues of a uniform $n \times n$ unitary matrix:



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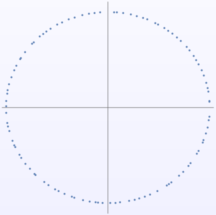

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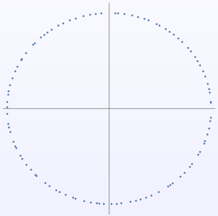

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Local picture: point process limit near any given point

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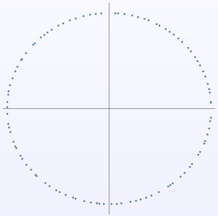

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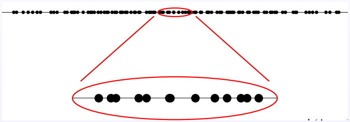
Limit is the same as the bulk limit of GUE

Point process limit



Finite n: spectrum of a random Hermitian/unitary matrix

zeros of the characteristic polynomial

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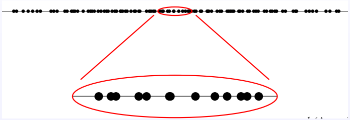

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Limit point process: spectrum of ??, zeros of ??

Quick detour to number theory

Riemann zeta function: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, for $\operatorname{Re} s > 1$.

(Analytic continuation to $\mathbb{C} \setminus \{1\}$)

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Is there a natural operator for the bulk limit of GUE?

Back to random matrices

Joint eigenvalue density for GUE and CUE:

$$\frac{1}{Z_n}\prod_{i<j\le n}|\lambda_j-\lambda_i|^2\prod_{i=1}^{n}e^{-\frac{1}{2}\lambda_i^2}, \qquad \frac{1}{Z_n'}\prod_{j<k\le n}|e^{-i\theta_j}-e^{i\theta_k}|^2.$$

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Many of the classical random matrix ensembles have joint eigenvalue densities of the form

$$\frac{1}{Z_{n,f,\beta}}\prod_{i<j\le n}|\lambda_j-\lambda_i|^{\beta}\prod_{i=1}^{n}f(\lambda_i)$$

with $\beta = 1, 2$ or 4 and f a specific reference density.

β-ensemble: finite point process with joint density

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$f(\cdot)$: reference density, $\beta > 0$

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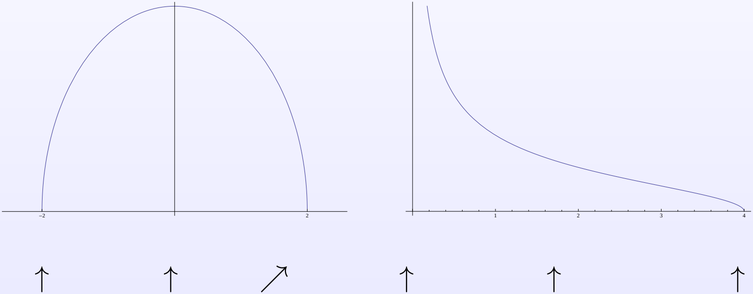
Examples:

- Hermite or Gaussian: normal density
- circular: uniform on the unit circle
- Laguerre or Wishart: gamma density
- Jacobi or MANOVA: beta density

$\beta = 1, 2, 4$: classical random matrix models

Scaling limits - global picture

Circular β-ensemble $\rightsquigarrow$ uniform law
Hermite β-ensemble $\rightsquigarrow$ semicircle law
Laguerre β-ensemble $\rightsquigarrow$ Marchenko-Pastur law



soft edge | bulk | s. e. | hard edge | bulk | s. e.

Point process limits

Soft edge: Rider-Ramírez-Virág (Hermite, Laguerre)
Airy_β process

Hard edge: Rider-Ramírez (Laguerre)
$\text{Bessel}_{\beta,a}$ processes

Bulk: Killip-Stoiciu, V.-Virág (circular, Hermite)
$C\beta E$ and Sine_β processes (later shown to be the same)

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Universality: Erdős-Yau-Bourgade, Krishnapur-Rider-Virág, Rider-Waters

Random operators

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Dumitriu-Edelman '02:

tridiagonal representation for Gaussian and Laguerre β-ensembles

Edelman-Sutton '06:

random tridiagonal matrices $\rightsquigarrow$ random differential operators

Conj.: Limit processes $\sim$ spectra of random differential operators

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Conj.: Limit processes $\sim$ spectra of random differential operators

Soft edge: Ramírez-Rider-Virág '06 (Gaussian, Laguerre)

$$\mathcal{A}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} dB$$

Hard edge: Ramírez-Rider '08 (Laguerre)

$$\mathcal{B}_{\beta,a} = -e^{(a+1)x + \frac{2}{\sqrt{\beta}} B(x)} \frac{d}{dx} \left\{ e^{-ax - \frac{2}{\sqrt{\beta}} B(x)} \frac{d}{dx} \right\}$$

B: standard Brownian motion, dB: white noise

domain: $[0, \infty) \to \mathbb{R}$, L^2 and boundary conditions

The Sine_β operator - operator in the bulk

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Thm (V-Virág '16):
There is a self-adjoint differential operator (Dirac-operator)

$$\tau : f \to 2R_t^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f'(t), \qquad f : [0,1) \to \mathbb{R}^2.$$

with spectrum given by the Sine_β process.

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Also: Several finite classical random matrix models, β-generalizations and scaling limits can be represented in this form.

Dirac operators

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R_t: positive definite matrix valued function

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Ingredients: a path $x_t + iy_t : [0, T) \to \mathbb{H}$ in the hyperbolic plane, two boundary points in $\mathbb{H}$.

$$X_t = \frac{1}{\sqrt{y_t}} \begin{bmatrix} 1 & -x_t \\ 0 & y_t \end{bmatrix}, \qquad R_t = X_t^T X_t.$$

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Two boundary points $\rightsquigarrow$ boundary conditions for τ

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Conjugation with X^{-1}: self-adjoint integral operator on L^2.

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- Sine_β (time-changed hyperbolic BM in $\mathbb{H}$)

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  time-change: $-\frac{4}{\beta}\log(1-t)$
- limits of certain one dimensional random Schrödinger operators (hyperbolic BM up to a fixed time)
- finite circular β-ensemble and circular Jacobi ensembles (random walk in $\mathbb{H}$)

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The function R_t and the corresponding path $x_t + iy_t$ are constant on each $[\frac{k}{n}, \frac{k+1}{n})$.

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Thm (Killip-Nenciu '04) If V is a uniformly chosen $n \times n$ unitary matrix then the Verblunsky coefficients (the parameters in the Szegő recursion) are independent with nice distributions.

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Construction of the hyperbolic RW: $b_0 = i, \dots, b_{n-1} \in \mathbb{H}$, $b_n \in \partial\mathbb{H}$

Given b_k we choose b_{k+1} uniformly on a hyperbolic circle with random radius ξ_k. In the Poincaré disk with center b_k we have $\xi_k^2 \sim \text{Beta}(1, \frac{\beta}{2}(n-k-1))$. The last step is chosen uniformly on $\partial\mathbb{H}$ as viewed from b_{n-1}.

Operator level bulk limit

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finite model

$$\downarrow$$

differential operator built from RW

$$\downarrow$$

integral operator built from RW

$$\downarrow$$

integral operator built from BM

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The previous methods required the derivation of a one-parameter family of SDE system.

Here we need to understand the limit of the integral kernel (convergence of a RW to a BM)

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Thm (V-Virág, '17):
One can couple the finite n circular β-ensembles to Sine_β so that the corresponding operators are within $\log^3 n \cdot n^{-1/2}$ in H-S norm.

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Coupling bound for $\beta = 2$: Maples, Najnudel, Nikeghbali ’13
TV bounds on the counting functions ($\beta = 2$): Meckes, Meckes ’16

Limits of characteristic polynomials

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Thm(Chhaibi, Najnudel, Nikeghbali '17): Label the points of Sine_2 as $\ldots < \lambda_{-1} < \lambda_0 < 0 < \lambda_1 < \ldots$ Then

$$\xi(z) := (1 - \tfrac{z}{\lambda_0}) \prod_{k=1}^{\infty} \left(1 - \tfrac{z}{\lambda_{-k}}\right) \left(1 - \tfrac{z}{\lambda_k}\right)$$

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$$\frac{p_n\left(e^{i\frac{z}{n}}\right)}{p_n(1)} \to e^{i\frac{z}{2}} \cdot \xi(z)$$

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general β?

$\beta = \infty$ case

The finite ensemble is just n equally spaced points on the circle, rotated with a uniform angle. The scaling limit is $2\pi\mathbb{Z} + U[0, 2\pi]$.

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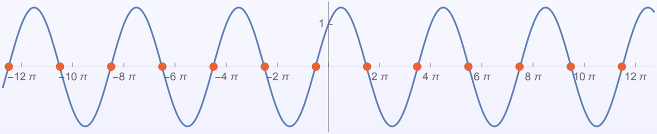

The limiting function is $\sin(z/2)$ with a random shift. After normalization:

$$\cos(z/2) + q\sin(z/2), \qquad q \sim \text{Cauchy}$$

Aizenmann-Warzel '15: On the ubiquity of the Cauchy distribution in spectral problems

Entire function from the random operator

$$\tau : f \to 2R_t^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f'(t), \qquad f : [0,1) \to \mathbb{R}^2.$$

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$\sum_k \frac{1}{\lambda_k^2} < \infty$ holds a.s. $\rightsquigarrow$ $\det_2(I - z\tau^{-1})$ is well defined

$$\det_2(I - z\tau^{-1}) = \prod_k (1 - z\lambda_k^{-1}) e^{z\lambda_k^{-1}}$$

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Thm(V., Virág): The scaling limit of the normalized characteristic polynomials for circular β-ensembles is given by

$$e^{i\frac{z}{2}} \cdot \det_2(I - z\tau^{-1}) e^{-z \cdot \text{"}\operatorname{Tr} \tau^{-1}\text{"}}$$

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The resulting function can be written as $A + qB$ where

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Using the scale invariance of the hyperbolic BM we can find an SPDE so that its stationary solution is $E = A - iB$:

$$dE_t = -i\frac{\beta}{8} z E_t(z) ds - \frac{\beta}{4} z \partial_z E_t(z) ds + \frac{\bar{E}_t(\bar{z}) - E_t(z)}{2i} dW, \quad E_0(z) = 1$$

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The SDE system for $\partial_z^n E_t(0), n = 1, 2, \ldots$ can be solved explicitly.

Moments of products of ratios

Borodin-Strahov '06: Limit of $E\left[\prod_{j=1}^{k}\frac{\tilde{p}_n(z_j)}{\tilde{p}_n(w_j)}\right]$ for various classical random matrix models.
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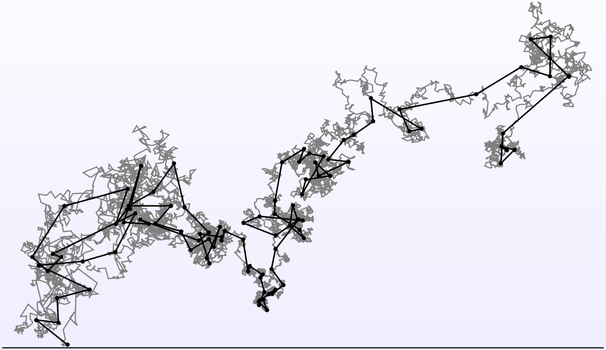
Outline: In the $\operatorname{Im} w_j < 0$ case $E\left[\prod_{j=1}^{k}\frac{A(z_j)+qB(z_j)}{A(w_j)+qB(w_j)}\right]$ can be expressed using $A - iB$. The expectation can now be evaluated using the SPDE representation for $A - iB$.



THANK YOU!