Recent results around long-range Ising models:
I) g versus Gibbs,
II) Inhomogeneous fields,
III) Metastates and random boundary conditions,
IV) Metastability
V) Interfaces

A. C. D. v.E.,
with R. Bissacot, L. Coquille, E. Endo, B. Kimura,
A. Le Ny, W.M. Ruszel, C.Spitoni

San Carlos, July, 2019

Long-range Ising models, mostly in $d = 1$, (Dyson models),
d-dimensional, long-range,
ferromagnetic Ising models with pair interactions.
Ising spins $\omega_i = \pm 1$.
Formally Hamiltonian
$H = \sum_{i,j \in Z^d} J(i-j)\omega_i \omega_j$.
with polynomial decay, e.g. $J(i-j) = -|i-j|^{-d\alpha}$.
Simulates high dimensions.
Varying decay power α between 1 and 2 is like
varying dimension in short-range models,
but possible in continuous way.
Slower decay corresponds to higher dimension.

Phase transitions possible, even in $d = 1$.
(Dyson, proving Kac-Thomson conjecture).
Different proofs since.
Use Cassandro-Ferrari-Merola-Presutti,
(plus Littin-Picco, plus Kimura).
"Contours, low-temperature expansion".

1) Approximately it holds
$\alpha \approx \frac{d+2}{d}$ for critical behaviour,
mean-field critical behaviour
for $\alpha < \frac{3}{2}$, like $d > 4$.
Suggestive, but only approximate guide.
2) For surface-to-volume arguments $\alpha \approx \frac{d+1}{d}$.
3) Never, for no α rigid interfaces in $d = 1$.
As in $d = 2$.

Energy estimates:
Flipping all spins in interval of length L
costs energy, boundary term,
maximally $O(L^{2-\alpha})$,
uniformly bounded energy when $\alpha > 2$.
Maximal energy between two half-lines is bounded.
Main ingredient for uniqueness (and analyticity, etc).

I) Topic I):
g versus Gibbs.
The difference between
Time and Space stochastic processes.
Time version:
(weak, continuous, dependence on the past.)
g-measures=
Chains of Infinite Order=
Chains with Complete Connections=
Uniform Martingales/Random Markov Processes.
(Keane 70's, Harris 50's,
Onicescu-Mihoc and Doeblin-Fortet 30's,
Kalikow 90's).
Studied in Ergodic Theory, Probability.

Spatial version:
Gibbs (=DLR) measures= Gibbs fields=
" almost" Markov random fields.
Discovered independently,
in East (mathematics)
and West (physics),
(Dobrushin, Lanford-Ruelle 60's).
Mathematical Physics.
Here two-state -Bernoulli- variables,
(= **Ising** spins:)
$\omega_i = \pm$, for all $i \in Z$.
(Can be much more general.)
Warning:
DLR Gibbs $\neq$ SRB Gibbs.

Gibbs measures:

Let G be an infinite graph, here Z.

Configuration space:

Space of sequences: $\Omega = \{-, +\}^G$.

Probability measures on Ω,

labeleled by **interactions**.

An interaction is a collection of functions,

$\Phi_X(\omega)$, dependent on $\{-, +\}^X$,

where the X are subsets of G.

Energy (**Hamiltonian**)
$H_\Lambda^{\Phi,\tau}(\omega) = \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
Sum of interaction-energy terms.

A measure μ is *Gibbs* iff:

(A version of) the

conditional probabilities of

finite-volume configurations,

given the outside configuration, satisfies:

$\mu(\omega_\Lambda|\tau_{\Lambda^c}) = \frac{1}{Z_\Lambda^\tau} \exp - \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.

for ALL

configurations ω,

boundary conditions τ

and finite volumes Λ.

Gibbsian form.
Rigorous version of
"$\mu = \frac{1}{Z} \exp -H$",
Gibbs canonical ensemble.
Larger energy means
exponentially smaller probability.
A Gibbs measure for a nearest-neighbour
model satisfies a
spatial Markov property:
$\mu(\omega_{\{1,\ldots,n\}}|\tau_{\{1\ldots.n\}^c}) = \mu(\omega_{\{1\ldots.n\}}|\tau_0\tau_{n+1})$.
Conditioned on the border spins,
at 0 and $n+1$,
inside and *outside* are independent.

A two-state Markov chain is again
a measure on the same sequence space Ω.
Now it has to satisfy the " ordinary"
(timelike) Markov property:
$\mu(\omega_{\{1...n\}}|\tau_{\{-\infty,....,0\}}) = \mu(\omega_{\{1...n\}}|\tau_0)$.
One can describe this via a product
of 2-by-2 stochastic matrices P
with non-zero entries:
$P(k,l) = P(\omega_i = k \rightarrow \omega_{i+1} = l)$.
Here $k, l = \pm$ and i is any site (=time) in Z.
Theorem: There is a one-to-one connection between
stationary (time-invariant)
2-state Markov Chains
and (space-translation-invariant) nearest-neighbor
Ising Gibbs measures.

Continuity (=almost Markov = quasilocality).
Product topology:
Two sequences are close if they are
equal on a large enough finite interval.
Topology metrisable, metric e.g. by:
$d(\omega, \omega') = 2^{-|n|}$,
where n is the site with
minimal distance from origin,
such that $\omega_n \neq \omega'_n$.
A function is continuous,
if it depends weakly on sites far away
and mostly on what happens not too far,
(or not too long ago)
whatever it is.

Processes (time) **g-measures**:
$\mu(\sigma_0 = \omega_0 | \omega_{Z-}) = g(\omega_0 \omega_{Z-})$,
with g-function continuous.
Probability of getting ω_0, given the past.
Continuous dependence on the **past**.
Continuity studied since the 30's
(Doeblin-Fortet).
Claim!?:
Continuity implies uniqueness (Harris(50's)).
Mistake in proof pointed out by Keane (70's).
Counterexamples due to Bramson-Kalikow (90's).
Sharper criterion Berger-Hoffman-Sidoravicius (2003-2017).

Random Fields (space) **Gibbs measures:**
Continuity of conditional probabilities
corresponds to summability of interactions.
$\sum_{0\in X} ||\Phi_X|| < \infty$.
Continuous dependence on **outside**
beyond the border. *(Quasilocality).*
No "Action at a distance".
(No observable influence from behind the moon)

Plus: "non-nullness".
Any finite change in the -infinite- system costs a finite amount of energy.
Any configuration in finite domain occurs with finite probability, whatever is happening outside.
Gibbs measures satisfy (equivalently) a finite-energy condition.
Equivalence holds (Kozlov-Sullivan):
Finite-energy $+$ continuity $=$ Gibbs.

Question:
Are g-measures and Gibbs measures equivalent notions?
Answer: No. Non-Gibbsian g-measures exist.
Fernández-Gallo-Maillard.
Our opposite Counterexample:
(Gibbs, non-g-measure).
Gibbs measures for Dyson models.
Low temperatures.
Long-range Ising models.
Ferromagnetic pair interactions.
$\Phi_{i,j}(\omega) = -J|i-j|^{-\alpha}\omega_i\omega_j$.
Interesting regime $1 < \alpha \leq 2$.
Phase transition for large J,
at low temperatures:

Two different Gibbs measures,
for the same interaction,
called μ^+ and μ^-, for such Φ.
Spatially continuous conditional probabilities.
Warning:
Phase transitions impossible for Markov Chains or Fields,
always uniqueness.

Claim:
At low T and for $\alpha^* < \alpha < 2$
Dyson Gibbs measures are not g-measures.
Here technical condition
$\alpha^* = 3 - \frac{\ln 3}{\ln 2}$.

Input:
Interface result for Dyson models
(Cassandro,Merola,Picco,Rozikov).
Take interval $[-L, +L]$,
all spins to the left are minus,
all spins to the right are plus.
Then there is an interface point **IF**, such that:
1) To the left of the interface
we are in the minus phase (μ^{-}),
to the right of the interface
we are in the plus phase (μ^{+}).
2) With overwhelming probability the location
of the interface is at most $O(L^{\frac{\alpha}{2}})$ from the center.
$... - - - - - m....|\mathbf{IF}| + m.....| + + + + + ...$

Observation 1:
If I change all spins left of a length-N interval of minuses, the effect from the left on the central O(L) interval is bounded by $O(LN^{1-\alpha})$, thus small for N large.

Consequence:
A large interval of minuses (size N) will have a moderately large (size L) interval of minus phase on both sides.
Interfaces are pushed away.

Observation 2:
If I decouple a comparatively small interval,
of size $L_1 = o(L)$,
in the beginning of my minus-phase interval,
this hardly changes the interface location.
(Cost of **IF** shift by εL is larger,
namely $O(L^{2-\alpha})$.
Shown by Cassandro et al.)

Observation 3:

If I make in this L_1 interval
an alternating configuration
+-+-+-+-+-...
then the total energy (influence)
on its complement
is bounded by the double sum

$\sum\sum_{i=1....L_1, j>L_1}(|j-i|^{-\alpha} - |j+1-i|^{-\alpha})=$
$\sum\sum_{i=1....L_1, j>L_1}(O(|j-i|^{-(\alpha+1)})=$
$\sum_{i=1....L_1} O(|i|^{-\alpha})$

which is bounded, uniformly in L_1.
Therefore finite, small effect.

Remark:
Effect only at positive temperature.
Entropic Repulsion.
A large alternating interval,
preceded by a MUCH
larger interval of minuses,
cannot shield the influence
of this homogeneous minus interval.
But this means precisely that
the conditional probability of finding a plus (or a minus),
at a given site, conditioned on an alternating past,
is not continuous.
Thus two-sided continuity
occurring at the same time
as one-sided discontinuity.

Conclusion of Topic I):

Two-sided continuous dependence

-spacelike- does *not* imply

one-sided continuous dependence

-timelike.

Topic II)
Inhomogeneous Fields.

Consider the Dyson model in an inhomogeneous field,
which decays to zero.
$H = \sum_{i,j \in Z} -J|i-j|^{-\alpha}\omega_i\omega_j - h_i\omega_i$.
Here the field decays to zero as a power:
$h_i = |i+1|^{-\gamma}$.
One can cook up a version of the "Imry-Ma" argument:
Boundary versus volume.
Energy cost of flipping (an interval) of length L around the origin
equals $O(L^{2-\alpha})$, (boundary, contour term)
energy gain due to following the field
equals $O(L^{1-\gamma})$ (volume term).
Which term dominates?

If $\gamma > \alpha - 1$, the "contour term" wins from the field,
the phase transition should persist,
in the opposite case $\gamma < \alpha - 1$,
the spins should follow the field,
and there should be a unique Gibbs measure.

Remarks:

1) Deterministic rather than random field.
2) Thermodynamically infinitesimal, smaller than each finite field.
3) Random decaying field, $h_i = X_i |i+1|^{-\gamma}$,
with X_i i.i.d. symmetric random variables,
intermediate case, energy due to field is $O(L^{\frac{1}{2}-\gamma})$,
similar arguments (Littin).
Standard Imry-Ma case (Cassandro-Orlandi-Picco)
occurs when $\gamma = 0$.

Proven:
Stability of transition for Dyson models.
Open:
Uniqueness regime.

(In short-range models
Bissacot-Cassandro-Cioletti-Presutti proved
both parts of the (deterministic)
Imry-Ma argument for decaying fields.)
In random fields Aizenman-Wehr proved uniqueness in $d = 2$,
Bricmont-Kupiainen persistence of transition in $d > 2$.

Topic III

Random boundary conditions and metastates.

Ferromagnets with random boundary conditions act to some degree as toy examples of quenched disordered systems.

They can display non-convergence

of the sequence of finite-volume measures in the thermodynamic limit (chaotic size dependence),

instead showing convergence in distribution to "metastates," distributions on Gibbs measures, which are (a subset of) the possible limit points (Newman-Stein, Aizenman-Wehr).

Concepts developed for Spin Glasses.

Consider a sufficiently sparse increasing sequence of intervals $\{-L_n, +L_n\}$ for Dyson models with random (Bernoulli) boundary conditions η, with $\eta_i = \pm$.

Question:

Then what could the limit points be of the sequence of finite-volume measures $\mu^{\alpha,\eta}_{L_N}$, when N diverges?

Answer:

Depends on α.

Case 1) Interaction across the boundary diverges when $\alpha < \frac{3}{2}$.

Like higher-dimensional short-range models.

Case 2) Boundary energy remains bounded otherwise.

Proof idea:

Let $W = \sum_{i<0, j>0} \eta_i |i - j^{-\alpha} = \sum_{i<0} |i|^{1-\alpha}$,
the interaction energy between
the plus configuration
on the positive half-line (Dyson ground state)
and a random configuration (boundary condition)
η on the negative half-line.
Then $EW = 0$, and
$EW^2 = \sum_{i<0} |i|^{2-2\alpha}$,
finite for $\alpha > \frac{3}{2}$,
infinite otherwise.

Consequences:
Case 1):
When $\alpha < \frac{3}{2}$,
the limit points are determined by the
sign of the diverging boundary term.
Thus one obtains
either the plus Gibbs measure μ^+
or the minus Gibbs measure μ^-.
The metastate Γ is the average of those two:
$\Gamma = \frac{1}{2}(\delta_{\mu^+} + \delta_{\mu^-})$.
Like higher-dimensional, $d > 1$, short-range Ising models.

Case 2):
When $\alpha > \frac{3}{2}$
New behaviour:
the boundary energies converge to some random variable.
Thus the sequence of finite-volume measures $\mu_{L_N}^{\alpha,\eta}$
now has as limit points Gibbs measures which are
mixtures of the plus and minus measures:
$\mu_\lambda = \lambda\mu^+ + (1-\lambda)\mu^-$.
As a consequence the metastate becomes
an average over these mixtures:
$\Gamma = \int P_\alpha(d\lambda)\mu_\lambda$,
with the mixture measure P_α
possibly dependent on the details of the model,
like the value of α.

Topic IV:
Metastability.

Again, boundary versus volume.

Consider the Dyson model in a weak field,
$H = -J\sum_{i,j\in Z} |i-j|^{-\alpha}\omega_i\omega_j - h\sum_{i\in Z}\omega_i$
with h small.

Ground state is $+$-configuration.

Take the metastable $-$ configuration.

Energy cost of inserting a $+$ interval of length L is $JL^{2-\alpha}$.

Energy gain due to field is $2hL$.

Thus length of critical droplet (interval)
$l_c = \frac{1}{1-\alpha}(\frac{J}{h})^{\frac{1}{1-\alpha}}$.

This can be used to perform
a dynamical nucleation analysis,
leading to estimates on mean exit times,
via the pathwise approach to metastability,
for volumes (intervals) which are
larger than the critical-droplet length..
In the limit where $\beta \to \infty$,
they behave like $\exp \mathbf{\Gamma}$,
with the energy barrier to cross
$\mathbf{\Gamma} = O(hl_c)$.

Topic V:
Interfaces.

Why are there no interface Gibbs measures
for Dyson models in $d = 1$?
Shifting the IF point by a finite distance
costs a finite amount of (free) energy.
But going from one extremal Gibbs measure to another one
MUST cost an infinite amount of free energy.
Extremal Gibbs measures cannot be
absolutely continuous with respect to each other,
they need be singular or the same.

So what happens in $d = 2$?
(Only interesting dimension.
In $d > 2$ even short-range models have
rigid interface Gibbs measures,
according to Dobrushin and van Beijeren).
In $d = 2$ no interface Gibbs measures
for short-range models (Aizenman-Higuchi).

Answer for long-range models:
Depends on **(an)isotropy**.
Case 1: isotropic case.
$H = \sum_{i,j \in Z^2} -|i-j|^{-\alpha} \omega_i \omega_j$
Long range: Decay power $2 < \alpha < 4$.
Summable and not short-range.
Low-T results:
No interface Gibbs measures.
Stronger proofs for $\alpha > 3$, via standard Peierls contours.
Reason. Maximal interaction between
half-line and half-plane is finite

But even for $\alpha > 2$:

Theorem:

Dobrushin boundary conditions do not produce a Dobrushin Gibbs measure.

Proof via relative entropy estimate.

Expected energy cost of shifting interface
by finite distance is finite.

Reason: Although a half-line can have an infinite
interaction with a half-plane,
shifting a Dobrushin interface costs only a finite
amount of energy, due to cancellations between the two sides of the interface.

Relative entropy between measure and its shift
equals expectation of difference between Hamiltonian and its shift:
$I(\mu|\mu T) = \mu(H - HT)$.
With Dobrushin boundary conditions antisymmetry,
between up and down.
Open Question:
Fluctuations of interface in L by L square?
Less than $\sqrt{L}$.
Which power?

Case 2:
Anisotropic case, interaction only along the axes:
$H = \sum_{i,j \in Z^2} |i_x - j_x|^{-\alpha_x} \omega_i \omega_j + \sum_{i,j \in Z^2} |i_y - j_y|^{-\alpha_y} \omega_i \omega_j$
Decay powers now satisfy $1 < min_{x,y} \alpha_{x,y} < 2$.
Then rigid interface Gibbs measures exist.
Proof follows van Beijeren.

Conclusions:
Long-range Ising models often behave like
higher-dimensional short-range models.
But this holds in **some** but **not all** respects.
Different behaviour of interfaces.
Used for finding Gibbs measure which is not g-measure.
Different behaviour of metastates.
Similar behaviour of metastability.
Similar behaviour in models in inhomogeneous fields.

I) With R. Bissacot (Sao Paulo), E. Endo (Shanghai),
A. Le Ny (Paris).
Comm. Math. Phys., 363, 767–788, 2018 (g vs Gibbs).
II) With R. Bissacot, E. Endo,
B. Kimura, W.M. Ruszel (Delft + Delft)
Ann. Henri Poincaré 19: 2557-2574, 2018 (Inhomogeneous fields).
III) With E. Endo and A. Le Ny (Metastates)
In progress.
(with Endo, Fernández, Verbitskiy, half-line analysis)
IV) With B. Kimura, W.M. Ruszel, C. Spitoni (Utrecht)
J. Stat. Phys. 174:1327–1345, 2019 (metastability).
V) With L. Coquille (Grenoble), A. Le Ny and W.M. Ruszel
J. Stat. Phys. 172: 1210-1222, 2018 (Interfaces).