# A Martingale Approach for Fractional Brownian Motions and Related Path-Dependent PDEs

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# Outline

# The standard risk neutral pricing

- Let $S$ be an underlying asset price, $\mathbb{P}$ a risk neutral measure :

$$dS_t = \sigma(t, S_t) dB_t$$

- Let $\xi = g(S_T)$ be a payoff at $T$, then the price at $t$ is :

$$Y_t = \boldsymbol{E}_t[\xi]$$

- In the above Markovian setting : $Y_t = u(t, S_t)$,

$$\partial_t u + \frac{1}{2}\sigma^2(t,x)\partial^2_{xx} u = 0, \quad u(T,x) = g(x).$$

- In path dependent setting : $\sigma = \sigma(t, S_\cdot), \xi = g(S_\cdot)$, then

$$Y_t = u(t, S_\cdot),$$
$$\partial_t u + \tfrac{1}{2}\sigma^2(t,\omega)\partial^2_{\omega\omega} u = 0, \quad u(T,\omega) = g(\omega).$$

# Rough volatility model

- Rough volatility : $dS_t = S_t\sigma_t dB_t$ and $\sigma$ is rough
  - ◇ See e.g. Gatheral-Jaisson-Rosenbaum (2014)
- A natural model : $\sigma$ driven by a fractional Brownian motion $B^H$
- Goal : characterize $Y_t := \boldsymbol{E}\big[\xi \mid \mathcal{F}_t^{B,B^H}\big]$
  - ◇ $\sigma$ (hence $B^H$) can be observed
  - ◇ To focus on the main idea we will assume $\xi$ is $\mathcal{F}_T^{B^H}$-measurable and consider $Y_t = \boldsymbol{E}\big[\xi \mid \mathcal{F}_t^{B^H}\big]$
  - ◇ Sone related recent works : El Euch-Rosenbaum (2017), Fouque-Hu (2017)

# Outline

1. Introduction
2. Heat equation
3. Functional Itô formula
4. Nonlinear extension

# Fractional Brownian Motion

- Let $B^H$ be a fBM with $0 < H < 1$ :
  - $\diamond$ $B^H_t - B^H_s \sim \text{Normal}(0, (t-s)^{2H})$
  - $\diamond$ $B^H = B$ when $H = \frac{1}{2}$
- Two main features :
  - $\diamond$ $B^H$ is not Markovian $(H \neq \frac{1}{2})$
  - $\diamond$ $B^H$ is not a semimartingale $(H < \frac{1}{2})$
- Our goal : characterize $Y_t := \boldsymbol{E}\left[g(B^H_\cdot)\, \middle|\mathcal{F}^{B^H}_t\right]$

# Heat equation in BM case

- Let $\xi := g(B_T)$ and $Y_t := \boldsymbol{E}_t[g(B_T)]$.

- Denote

$$v(t,x) := \boldsymbol{E}\Big[g(x + B_T - B_t)\Big] = \int_{\mathbb{R}} g(y)p(T-t, y-x)dy$$

$$\text{where} \quad p(t,x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

- Heat equation :

$$\partial_t p(t,x) - \tfrac{1}{2}\partial_{xx} p(t,x) = 0$$

$$\partial_t v(t,x) + \tfrac{1}{2}\partial_{xx} v(t,x) = 0, \quad v(T,x) = g(x).$$

- $Y_t = v(t, B_t)$, $0 \le t \le T$

# A Heat equation for fBM

- Let $\xi := g(B^H_T)$ and $Y_t := \boldsymbol{E}_t[g(B^H_T)]$.
- Denote

$$v(t,x) := \boldsymbol{E}\left[g(x + B^H_T - B^H_t)\right] = \int_{\mathbb{R}} g(y) p_H(T-t, y-x) dy$$

$$\text{where} \quad p_H(t,x) := \frac{1}{\sqrt{2\pi} t^H} e^{-\frac{x^2}{2t^{2H}}}.$$

- Heat equation :

$$\partial_t v(t,x) + H t^{2H-1} \partial_{xx} v(t,x) = 0, \quad v(T,x) = g(x).$$

- $Y_0 = v(0,0)$

# A heat equation for fBM

- Let $\xi := g(B^H_T)$ and $Y_t := \boldsymbol{E}_t[g(B^H_T)]$.
- Denote $v(t,x) := \boldsymbol{E}\big[g(x + B^H_T - B^H_t)\big]$
- Heat equation :

$$\partial_t v(t,x) + Ht^{2H-1}\partial_{xx} v(t,x) = 0, \quad v(T,x) = g(x).$$

- $Y_0 = v(0, B^H_0),\ \ Y_T = v(T, B^H_T)$
- However, $v(t, B^H_t)$ is not a martingale :

$$Y_t \neq v(t, B^H_t) \text{ for } 0 < t < T.$$

# A crucial representation of fBM

- Representation : $B^H_t = \int_0^t K(t,r)dW_r$
  - $\diamond$ $\mathbb{F} := \mathbb{F}^{B^H} = \mathbb{F}^W$
  - $\diamond$ $K(t,r) \sim (t-r)^{2H-1}$, which blows up at $t=r$ when $H < \frac{1}{2}$

- Decomposition :

$$B^H_T = \int_0^T K(T,r)dW_r = \int_0^t K(T,r)dW_r + \int_t^T K(T,r)dW_r$$

- $\diamond$ $\int_0^t K(T,r)dW_r$ is $\mathcal{F}_t$-measurable
- $\diamond$ $\int_t^T K(T,r)dW_r$ is independent of $\mathcal{F}_t$
- $\diamond$ The previous decomposition $B^H_T = B^H_t + [B^H_T - B^H_t]$ does not satisfy this property

# An alternative heat equation

- Let $\xi := g(B^H_T)$ and

$$Y_t = \boldsymbol{E}_t\Big[g\Big(\int_0^t K(T,r)dW_r + \int_t^T K(T,r)dW_r\Big)\Big]$$

- Denote $v(t,x) := \boldsymbol{E}\Big[g\Big(x + \int_t^T K(T,r)dW_r\Big)\Big]$

- Then $Y_t = v\Big(t, \int_0^t K(T,r)dW_r\Big), 0 \le t \le T$

- Note : $v\Big(t, \int_0^t K(T,r)dW_r\Big)$ is a martingale

- Heat equation :

$$\partial_t v(t,x) + \frac{1}{2}K^2(T,t)\partial_{xx} v(t,x) = 0, \quad v(T,x) = g(x).$$

# A closer look

- $\Theta_T^t := \int_0^t K(T,r)dW_r = \boldsymbol{E}_t[B_T^H]$ is $\mathcal{F}_t$-measurable
  - $\diamond$ $\Theta_T^t$ is the forward variance and is observable in market
- Three ways to express $Y_t$ :

$$Y_t = v_1(t, B^H_{t\wedge\cdot}) = v_2(t, W_{t\wedge\cdot}) = v(t, \Theta_T^t)$$

  - $\diamond$ $B^H$ is not a semimartingale
  - $\diamond$ $W$ is a martingale (of course) but $v_2$ is not continuous
  - $\diamond$ $v$ has desired regularity and $t \mapsto \Theta_T^t$ is a martingale

## An extension

- Denote $Y_t := \boldsymbol{E}_t\Big[g(B_T^H) + \int_t^T f(s, B_s^H)ds\Big]$.
- By previous computation :

$$
\begin{aligned}
Y_t &= \boldsymbol{E}_t[g(B_T^H)] + \int_t^T \boldsymbol{E}_t[f(s, B_s^H)]ds \\
&= v(T, g; t, \boldsymbol{E}_t[B_T^H]) + \int_t^T v(s, f(s,\cdot); t, \boldsymbol{E}_t[B_s^H])ds \\
&= u(t, \{\boldsymbol{E}_t[B_s^H]\}_{t\le s\le T})
\end{aligned}
$$

- Note : $u$ is path dependent
  - ◇ If $H = \frac{1}{2}$, $\boldsymbol{E}_t[B_s] = B_t$, so $Y_t = u(t, B_t)$ is state dependent
  - ◇ In more general cases,

$$Y_t = u\Big(t, \{B_s^H\}_{0\le s\le t} \otimes_t \{\boldsymbol{E}_t[B_s^H]\}_{t\le s\le T}\Big).$$

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# The canonical setup

- Recall

$$Y_t = u\Big(t, \{B^H_s\}_{0\le s\le t} \otimes_t \{\boldsymbol{E}_t[B^H_s]\}_{t\le s\le T}\Big).$$

- For $t \in [0, T]$, $\omega \in \mathbb{D}^0([0,t))$, and $\theta \in C^0([t,T])$, define :

$$(\omega \otimes_t \theta)_s := \omega_s \mathbf{1}_{[0,t)}(s) + \theta_s \mathbf{1}_{[t,T]}(s), \quad 0 \le s \le T.$$

- The canonical space :

$$\Lambda := \Big\{(t, \omega \otimes_t \theta) : t \in [0, T], \omega \in \mathbb{D}^0([0,t)), \theta \in C^0([t,T])\Big\};$$

$$\Lambda_0 := \Big\{(t, \omega \otimes_t \theta) \in \Lambda : \omega \in C^0([0,t]), \omega_0 = 0, \theta_t = \omega_t\Big\}.$$

# Continuous mapping

- Recall

$$\Lambda := \Big\{(t, \omega \otimes_t \theta) : t \in [0, T], \omega \in \mathbb{D}^0([0, t)), \theta \in C^0([t, T])\Big\}.$$

- The metric :

$$\mathbf{d}((t, \omega \otimes_t \theta), (t', \omega' \otimes_{t'} \theta'))$$

$$:= \sqrt{|t - t'|} + \sup_{0 \le s \le T} \big|(\omega \otimes_t \theta)_s - (\omega' \otimes_{t'} \theta')_s\big|.$$

- $C^0(\Lambda)$ : continuous mapping $u : \Lambda \to \mathbb{R}$
- $C^0_b(\Lambda)$ : bounded $u \in C^0(\Lambda)$

# Path derivatives

- Time derivative :

$$\partial_t u(t, \omega \otimes_t \theta) := \lim_{\delta \downarrow 0} \frac{u(t+\delta, \omega \otimes_t \theta) - u(t, \omega \otimes_t \theta)}{\delta}.$$

◇ $\partial_t u$ is the right time derivative !

- First order spatial derivative : Fréchet derivative with respect to $\theta$

$$\langle \partial_\theta u(t, \omega \otimes_t \theta), \eta \rangle := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big[ u(t, \omega \otimes_t (\theta + \varepsilon \eta)) - u(t, \omega \otimes_t \theta) \Big],$$

for all $(t, \omega \otimes_t \theta) \in \Lambda$, $\eta \in C^0([t, T])$.

# Path derivatives (cont)

- Second order spatial derivative : bilinear operator on $C^0([t, T])$ :

$$\langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (\eta_1, \eta_2) \rangle$$

$$:= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big[ \langle \partial_\theta u(t, \omega \otimes_t (\theta + \varepsilon \eta_1)), \eta_2 \rangle - \langle \partial_\theta u(t, \omega \otimes_t \theta), \eta_2 \rangle \Big].$$

for all $(t, \omega \otimes_t \theta) \in \Lambda$, $\eta_1, \eta_2 \in C^0([t, T])$.

- Define the spaces $C^{1,2}(\Lambda)$ and $C^{1,2}_b(\Lambda)$ in obvious sense

## Functional Ito formula : $H \geq \frac{1}{2}$

- Regular case : $K(t,t)$ is finite and thus

$$s \in [t,T] \mapsto K_s^t := K(s,t) \text{ is in } C^0([t,T]).$$

- Denote : $X_s := B_s^H,\ 0 \leq s \leq t\,;\quad \Theta_s^t := \boldsymbol{E}_t[B_s^H],\ t \leq s \leq T$

- Functional Ito formula :

$$\begin{aligned} & du(t, X \otimes_t \Theta^t) \\ = \quad & \partial_t u(\cdot)dt + \langle \partial_\theta u(\cdot), K^t \rangle dW_t + \frac{1}{2}\langle \partial^2_{\theta\theta} u(\cdot), (K^t, K^t)\rangle dt. \end{aligned}$$

$\diamond$ If $H = \frac{1}{2}$, $K = 1$, this is exactly Dupire's functional Ito formula

## Functional Ito formula : $H < \frac{1}{2}$

- $K(s,t) \sim (s-t)^{H-\frac{1}{2}}$, $\partial_s K(s,t) \sim (s-t)^{H-\frac{3}{2}}$, $0 \le t < s \le T$
- For some $\alpha > \frac{1}{2} - H$, for any $(t, \omega \otimes_t \theta) \in \Lambda_0$, any $t < t_1 < t_2 \le T$, any $\eta \in C^0([t,T]$ with support in $[t_1, t_2]$,

$$\langle \partial_\theta u(t, \omega \otimes_t \theta), \eta \rangle \le C[t_2 - t_1]^\alpha \|\eta\|_\infty,$$

$$\langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (\eta, \eta) \rangle \le C[t_2 - t_1]^{2\alpha} \|\eta\|^2_\infty.$$

  ◇ Roughly speaking, we want $\partial_{\theta_t} u(t, \omega \otimes_t \theta) = 0$.

- Denote $K^{t,\delta}_s := K^t_{(t+\delta)\vee s}$. Then the following limits exist :

$$\langle \partial_\theta u(t, \omega \otimes_t \theta), K^t \rangle := \lim_{\delta \to 0} \langle \partial_\theta u(t, \omega \otimes_t \theta), K^{t,\delta} \rangle;$$

$$\langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (K^t, K^t) \rangle := \lim_{\delta \to 0} \langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (K^{t,\delta}, K^{t,\delta}) \rangle.$$

- Functional Ito formula still holds

# Linear path-dependent PDE

- $Y_t := \boldsymbol{E}_t\Big[g(B^H_T) + \int_t^T f(s, B^H_s)ds\Big] = u(t, X \otimes_t \Theta^t)$

- $Y_t + \int_0^t f(s, B^H_s)ds$ is a martingale

- Linear PPDE :

$$\partial_t u(t,\omega \otimes_t \theta) + \frac{1}{2}\langle \partial^2_{\theta\theta} u(t,\omega \otimes_t \theta), (K^t, K^t)\rangle + f(t,\omega_t) = 0,$$
$$u(T,\omega) = g(\omega_T).$$

- Theorem. Assume $f$ and $g$ are smooth, then the above PPDE has a unique classical solution $u$.

# Outline

## Nonlinear dynamics

- Forward dynamics : Volterra SDE

$$X_t = x + \int_0^t b(t; r, X_\cdot)dr + \int_0^t \sigma(t; r, X_\cdot)dW_r$$

- Backward dynamics : BSDE

$$Y_t = g(X_\cdot) + \int_t^T f(s, X_\cdot, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

$\diamond$ The backward one itself is time consistent. If we consider Volterra type of BSDEs, see a series of works by Jiongmin Yong.

- $Y_t = u(t, X \otimes_t \Theta^t)$, where

$$\Theta^t_s := x + \int_0^t b(s; r, X_\cdot)dr + \int_0^t \sigma(s; r, X_\cdot)dW_r, \ t \le s \le T.$$

# Nonlinear PPDE

- Representation : $u(t,\omega\otimes_t\theta) := Y_t^{t,\omega\otimes_t\theta}$, where

$$X_s^{t,\omega\otimes_t\theta} = \theta_s + \int_t^s b(s;r,\omega\otimes_t X_\cdot^{t,\omega\otimes_t\theta})dr + \int_t^s \sigma(s;r,\omega\otimes_t X_\cdot^{t,\omega\otimes_t\theta})dW_r$$

$$Y_s^{t,\omega\otimes_t\theta} = g(\omega\otimes_t X_\cdot^{t,\omega\otimes_t\theta}) - \int_s^T Z_r^{t,\omega\otimes_t\theta}dW_r + \int_s^T f(r,\omega\otimes_t X_\cdot^{t,\omega\otimes_t\theta}, Y_r^{t,\omega\otimes_t\theta}, Z_r^{t,\omega\otimes_t\theta})dr.$$

- Semilinear PPDE : $\varphi_s^{t,\omega} := \varphi(s;t,\omega)$, $t\le s\le T$, for $\varphi = b,\sigma$,

$$\partial_t u + \frac{1}{2}\langle\partial^2_{\theta\theta}u, (\sigma^{t,\omega},\sigma^{t,\omega})\rangle + \langle\partial_\theta u, b^{t,\omega}\rangle + f\left(t,\omega,u,\langle\partial_\theta u,\sigma^{t,\omega}\rangle\right) = 0,$$

$$u(T,\omega) = g(\omega).$$

## Further research

- Controlled problems (fully nonlinear PPDE)
- Viscosity solution
- Efficient numerical algorithms

# Pricing in a rough Heston model

- The model (El Euch and Rosenbaum 2017) : $cor(B, W) = \rho$,

$$S_t = S_0 + \int_0^t S_r \sqrt{V_r} dB_r;$$
$$V_t = V_0 + \frac{1}{\Gamma} \int_0^t (t-r)^{H-\frac{1}{2}} \Big[\lambda[\theta - V_r]dr + \nu\sqrt{V_r}dW_r\Big]$$

- Pricing : $Y_t = \boldsymbol{E}[g(S_T)|\mathcal{F}_t^{S,V}] = u(t, S_t, \Theta^t_{[t,T]})$

$$\Theta^t_s = V_0 + \frac{1}{\Gamma} \int_0^t (s-r)^{H-\frac{1}{2}} \Big[\lambda[\theta - V_r]dr + \nu\sqrt{V_r}dW_r\Big]$$

- $u$ satisfies certain path dependent PDE

## Hedging in the rough Heston model

- Replicability of $\Theta^t_s$ :

$$\Theta^t_s = \boldsymbol{E}_t[V_s] - \frac{1}{\Gamma}\int_t^s (s-r)^{H-\frac{1}{2}}\lambda[\theta - \boldsymbol{E}_t[V_r]]dr.$$

- Hedging : $a^t_s := (s-t)^{H-\frac{1}{2}}$,

$$\begin{aligned} dY_t &= \partial_x u\big(t, S_t, \Theta^t_{[t,T]}\big) dS_t \\ &\quad + (T-t)^{\frac{1}{2}-H}\Big\langle \partial_\theta u\big(t, S_t, \Theta^t_{[t,T]}\big), a^t\Big\rangle \times \\ &\qquad \Big[d\boldsymbol{E}_t[V_T] + \frac{1}{\Gamma}\int_0^T (T-r)^{H-\frac{1}{2}}\lambda d\boldsymbol{E}_t[V_r]dr\Big]. \end{aligned}$$

the derivative $g(S_T)$ by using the stock $S$ and the forward variance $\hat{\Theta}$. The hedging portfolio relies on the Frechet derivative of $C_t$ and certain characteristic functions, which requires the special structure of (5.1) and that $C_T = g(S_T)$ is state dependent.

We now explain how our framework covers the above example and beyond. First note that, for $X = (S, V)^\top$, (5.1) is a Volterra SDE (3.1) with

$$
\begin{aligned}
b(t; r, x_1, x_2) &= \begin{bmatrix} 0 \\ \frac{\lambda(t-r)^{H-\frac{1}{2}}[\theta - x_2]}{\Gamma(H+\frac{1}{2})} \end{bmatrix}, \\
\sigma(t; r, x_1, x_2) &= \begin{bmatrix} \sqrt{1-\rho^2} x_1 \sqrt{x_2} & \rho x_1 \sqrt{x_2} \\ 0 & \frac{\nu(t-r)^{H-\frac{1}{2}}\sqrt{x_2}}{\Gamma(H+\frac{1}{2})} \end{bmatrix}.
\end{aligned}
\tag{5.4}
$$

One may easily check that (5.1) satisfies all the properties in Assumptions 3.1 and 3.15, needed in Section 3.3 for $H \in (0, 1/2)$, see Remark A.2 below. Note that the dynamics of $S$ is standard, without involving a two-time-variable kernel. While we may apply the results in previous sections directly on the two dimensional SDE (5.1), for simplicity we restrict the path dependence only to the dynamics of $V$. Therefore, recall (2.6), for $t < s$ we denote

$$
\Theta^t_s := V_0 + \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \int_0^t (s-r)^{H-\frac{1}{2}} \left[\lambda[\theta - V_r]dr + \nu \sqrt{V_r} dW^2_r\right].
\tag{5.5}
$$

By the special structure of the rough Heston model, we can actually see that

$$
C_t = u\big(t, S_t, \Theta^t_{[t,T]}\big).
\tag{5.6}
$$

In particular, the dependence of $C_t$ on $S$ is only via $S_t$ and its dependence on $V$ does not involve $V_{[0,t)}$. Denote $u$ as $u(t, x, \omega)$ and we shall assume $g$ is smooth which would imply the smoothness of $u$. Now following the arguments in Section 4.1, in particular noting that $C$ is a martingale, we see that $u$ satisfies the following PPDE:

$$
\begin{aligned}
&\partial_t u + \frac{\lambda[\theta - \omega_t]}{\Gamma(H+\frac{1}{2})} \langle \partial_\omega u, a^t \rangle + \frac{x^2 \omega_t}{2} \partial^2_{xx} u + \frac{\rho \nu x \omega_t}{\Gamma(H+\frac{1}{2})} \langle \partial_\omega(\partial_x u), a^t \rangle \\
&+ \frac{\nu^2 \omega_t}{2\Gamma(H+\frac{1}{2})} \langle \partial^2_{\omega\omega} u, (a^t, a^t) \rangle = 0, \qquad \text{where} \quad a^t_s := (s-t)^{H-\frac{1}{2}}.
\end{aligned}
\tag{5.7}
$$

Moreover, by Theorem 3.17, we have (recalling $V_t = \Theta^t_t$)

$$
dC_t = \partial_x u\big(t, S_t, \Theta^t_{[t,T]}\big) dS_t + \frac{\nu \sqrt{V_t}}{\Gamma(H+\frac{1}{2})} \Big\langle \partial_\omega u\big(t, S_t, \Theta^t_{[t,T]}\big), a^t \Big\rangle dW^2_t.
\tag{5.8}
$$

The first term in the right side above obviously provides the $\Delta$-hedging in terms of the stock $S$. Note further that $t \mapsto \Theta_T^t$ is a semi-martingale, and we have

$$\frac{\nu\sqrt{V_t}}{\Gamma(H+\frac{1}{2})}dW_t^2 = (T-t)^{\frac{1}{2}-H}d\Theta_T^t \; - \; \frac{\lambda[\theta - V_t]}{\Gamma(H+\frac{1}{2})}dt.$$

Then

$$\begin{aligned} dC_t &= \partial_x u\big(t, S_t, \Theta_{[t,T]}^t\big)dS_t + (T-t)^{\frac{1}{2}-H}\Big\langle \partial_\omega u\big(t, S_t, \Theta_{[t,T]}^t\big), a^t\Big\rangle d\Theta_T^t \\ &\quad - \frac{\lambda[\theta - V_t]}{\Gamma(H+\frac{1}{2})}\Big\langle \partial_\omega u\big(t, S_t, \Theta_{[t,T]}^t\big), a^t\Big\rangle dt. \end{aligned} \tag{5.9}$$

That is, provided that we could replicate $\Theta_T^t$ using market instruments, which we will discuss in details below, then we may (perfectly) hedge $g(S_T)$ as claimed in [20].

We note that our $\Theta^t$ in (5.5) is different from the forward variance $\hat{\Theta}^t$ in (5.3). However, it can easily be replicated by using $\hat{\Theta}^t$, which can further be replicated (approximately) by variance swaps. Indeed, by (5.5) and taking conditional expectation on the dynamics of $V$ in (5.1), we see that

$$\hat{\Theta}_s^t = \Theta_s^t + \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\int_t^s (s-r)^{H-\frac{1}{2}}\lambda[\theta - \hat{\Theta}_r^t]dr, \quad t \le s \le T. \tag{5.10}$$

For any fixed $t$, clearly $\Theta_s^t$ is uniquely determined by $\{\hat{\Theta}_r^t\}_{t\le r\le s}$:

$$\Theta_s^t = \hat{\Theta}_s^t - \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\int_t^s (s-r)^{H-\frac{1}{2}}\lambda[\theta - \hat{\Theta}_r^t]dr. \tag{5.11}$$

In particular, this implies that, provided we observe the forward variance $\hat{\Theta}_s^t$, the process $\Theta_s^t$ is also observable at $t$. Moreover, as a function of $t$,

$$d\Theta_T^t = d\hat{\Theta}_T^t + \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}(T-t)^{H-\frac{1}{2}}\lambda[\theta - \hat{\Theta}_t^t]dt.$$

Plug this into (5.9) and note that $\hat{\Theta}_t^t = V_t$, we obtain

$$dC_t = \partial_x u\big(t, S_t, \Theta_{[t,T]}^t\big)dS_t + (T-t)^{\frac{1}{2}-H}\big\langle \partial_\omega u\big(t, S_t, \Theta_{[t,T]}^t\big), a^t\big\rangle d\hat{\Theta}_T^t. \tag{5.12}$$

That is, $C_T$ can be replicated by using $S_t$ and $\hat{\Theta}_T^t$, with the corresponding hedging portfolios $\partial_x u$ and $(T-t)^{\frac{1}{2}-H}\big\langle \partial_\omega u, a^t\big\rangle$, respectively.