Convergence of symmetric Feller processes on metric trees

Anita Winter, University of Duisburg-Essen

based on joint work with **Siva Athreya** and **Wolfgang Löhr**

"XXIII Escola Brasileira de Probabilidade"

São Carlos, July 22$^{\mathrm{nd}}$– 27$^{\mathrm{th}}$ 2019

UNIVERSITÄT DUISBURG ESSEN

Offen im Denken

Example I: Symmetric random walk on $\mathbb{Z}$

- Consider the **symmetric RW** (in discrete time) $S := (S_n)_{n\geq 0}$ on $\mathbb{Z}$, i.e., the MC with transition probabilities

$$p(k, k \pm 1) = \frac{1}{2}$$

for all $k \in \mathbb{Z}$.

Example I: Symmetric random walk on $\mathbb{Z}$

- Consider the **symmetric RW** (in discrete time) $S := (S_n)_{n\geq 0}$ on $\mathbb{Z}$, i.e., the MC with transition probabilities

$$p(k, k\pm 1) = \frac{1}{2}$$

for all $k \in \mathbb{Z}$.

- Put for all $m \in \mathbb{N}$,

$$\tau_{\{-m,m\}} := \inf\{n \geq 0 : \; S_n \in \{-m, m\}\}.$$

Then by optional sampling, $\mathbb{E}_0[\tau_{\{-m,m\}}] = \mathbb{E}_0[S^2_{\tau_{-m,m}}] = m^2$, which suggests the *Brownian rescaling*.

Example I: Symmetric random walk on $\mathbb{Z}$

- Consider the **symmetric RW** (in discrete time) $S := (S_n)_{n\geq 0}$ on $\mathbb{Z}$, i.e., the MC with transition probabilities

$$p(k, k\pm 1) = \frac{1}{2}$$

for all $k \in \mathbb{Z}$.

- Put for all $m \in \mathbb{N}$,

$$\tau_{\{-m,m\}} := \inf\{n \geq 0 :\ S_n \in \{-m, m\}\}.$$

Then by optional sampling, $\mathbb{E}_0[\tau_{\{-m,m\}}] = \mathbb{E}_0[S^2_{\tau_{-m,m}}] = m^2$, which suggests the *Brownian rescaling*.

- Indeed, the *functional CLT* holds:

$$\left(\tfrac{1}{m} S_{\lfloor m^2 t\rfloor}\right)_{t\geq 0} \underset{m\to\infty}{\Longrightarrow} \left(B_t\right)_{t\geq 0},$$

weakly in path space.

Example I: Symmetric random walk with drift on $\mathbb{Z}$

- Consider for each $m \in \mathbb{N}$ the RW $S^{(m)} := (S_n^{(m)})_{n \geq 0}$ on $\mathbb{Z}$ **with small drift** $c_m := \frac{c}{m}$, $c > 0$, i.e., the MC with transition probabilities

$$p^{(m)}(k, k \pm 1) = \frac{1}{2}(1 \pm c_m)$$

for all $k \in \mathbb{Z}$.

Example I: Symmetric random walk with drift on $\mathbb{Z}$

- Consider for each $m \in \mathbb{N}$ the RW $S^{(m)} := (S_n^{(m)})_{n\geq 0}$ on $\mathbb{Z}$ **with small drift** $c_m := \frac{c}{m}$, $c > 0$, i.e., the MC with transition probabilities

$$p^{(m)}(k, k \pm 1) = \frac{1}{2}(1 \pm c_m)$$

for all $k \in \mathbb{Z}$.

- Put for all $m \in \mathbb{N}$, $\tau_{\{-m,m\}} := \inf\{n \geq 0 : \; S_n \in \{-m, m\}\}$. Once more by the optional sampling,

$$\mathbb{E}_0^{(m)}\left[\tau_{\{-m,m\}}\right] = \mathcal{O}(m^2).$$

Example I: Symmetric random walk with drift on $\mathbb{Z}$

- Consider for each $m \in \mathbb{N}$ the RW $S^{(m)} := (S_n^{(m)})_{n\geq 0}$ on $\mathbb{Z}$ **with small drift** $c_m := \frac{c}{m}$, $c > 0$, i.e., the MC with transition probabilities

$$p^{(m)}(k, k \pm 1) = \frac{1}{2}(1 \pm c_m)$$

for all $k \in \mathbb{Z}$.

- Put for all $m \in \mathbb{N}$, $\tau_{\{-m,m\}} := \inf\{n \geq 0 : \ S_n \in \{-m, m\}\}$. Once more by the optional sampling,

$$\mathbb{E}_0^{(m)}\big[\tau_{\{-m,m\}}\big] = \mathcal{O}(m^2).$$

- The *functional CLT* reads now:

$$\big(\tfrac{1}{m} S^{(m)}_{\lfloor m^2 t \rfloor}\big)_{t\geq 0} \underset{m\to\infty}{\Longrightarrow} \big(B_t + ct\big)_{t\geq 0},$$

weakly in path space.

Motivation II: Sinai's RWRE on $\mathbb{Z}$

- Consider a family $\omega = (\omega_z^-)_{z\in\mathbb{Z}}$ of i.i.d. $(0,1)$-valued r.v.

Motivation II: Sinai's RWRE on $\mathbb{Z}$

- Consider a family $\omega = (\omega_z^-)_{z\in\mathbb{Z}}$ of i.i.d. $(0,1)$-valued r.v.
- Let $X := ((X_n)_{n\in\mathbb{N}}, \mathbf{P}_\omega^z, z \in \mathbb{Z})$ be the **RWRE** on $\mathbb{Z}$, i.e., the MC that given a realization ω has transition probabilities

$$p^\omega\big(z, z \pm 1\big) = \omega_z^\pm \quad \text{with} \;\; \omega_z^+ := 1 - \omega_z^- .$$

Motivation II: Sinai's RWRE on $\mathbb{Z}$

- Consider a family $\omega = (\omega_z^-)_{z\in\mathbb{Z}}$ of i.i.d. $(0,1)$-valued r.v.
- Let $X := ((X_n)_{n\in\mathbb{N}}, \mathbf{P}_\omega^z, z \in \mathbb{Z})$ be the **RWRE** on $\mathbb{Z}$, i.e., the MC that given a realization ω has transition probabilities

$$p^\omega\big(z, z \pm 1\big) = \omega_z^\pm \quad \text{with} \;\; \omega_z^+ := 1 - \omega_z^-.$$

- Put $\rho_z := \omega_z^- / \omega_z^+$, and assume
  - *Recurrence.* $E[\log \rho_0] = 0$ with $\sigma := \mathrm{Var}(\log \rho_0) > 0$.
  - *Uniform ellipticity.* $P(\{-\epsilon < \rho_0 \le 1 - \epsilon\}) = 1$ for $\epsilon \in (0, \frac{1}{2})$.

Motivation II: Sinai's RWRE on $\mathbb{Z}$

- Consider a family $\omega = (\omega_z^-)_{z\in\mathbb{Z}}$ of i.i.d. $(0,1)$-valued r.v.
- Let $X := ((X_n)_{n\in\mathbb{N}}, \mathbf{P}_\omega^z, z \in \mathbb{Z})$ be the **RWRE** on $\mathbb{Z}$, i.e., the MC that given a realization ω has transition probabilities

$$p^\omega(z, z \pm 1) = \omega_z^\pm \quad \text{with} \;\; \omega_z^+ := 1 - \omega_z^- .$$

- Put $\rho_z := \omega_z^- / \omega_z^+$, and assume
    - *Recurrence.* $E[\log \rho_0] = 0$ with $\sigma := \mathrm{Var}(\log \rho_0) > 0$.
    - *Uniform ellipticity.* $P(\{-\epsilon < \rho_0 \le 1 - \epsilon\}) = 1$ for $\epsilon \in (0, \frac{1}{2})$.
- *annealed (weak) LLN; [Sinai (1982)]* There is a r.v. B s.t. for all $\eta > 0$,

$$\int \mathbf{P}_\omega^z\big(\{\big|\tfrac{\sigma^2 X_n}{(\log n)^2} - B\big| > \eta\}\big)\, P(\mathrm{d}\omega) \underset{n\to\infty}{\longrightarrow} 0.$$

Motivation II: Sinai's RWRE on $\mathbb{Z}$

- Consider a family $\omega = (\omega_z^-)_{z\in\mathbb{Z}}$ of i.i.d. $(0,1)$-valued r.v.
- Let $X := ((X_n)_{n\in\mathbb{N}}, \mathbf{P}^z_\omega, z\in\mathbb{Z})$ be the **RWRE** on $\mathbb{Z}$, i.e., the MC that given a realization ω has transition probabilities
$$p^\omega(z, z\pm 1) = \omega_z^\pm \quad \text{with } \ \omega_z^+ := 1-\omega_z^-.$$
- Put $\rho_z := \omega_z^-/\omega_z^+$, and assume
  - *Recurrence.* $E[\log\rho_0] = 0$ with $\sigma := \mathrm{Var}(\log\rho_0) > 0$.
  - *Uniform ellipticity.* $P(\{-\epsilon < \rho_0 \le 1-\epsilon\}) = 1$ for $\epsilon\in(0,\frac{1}{2})$.
- *annealed (weak) LLN; [Sinai (1982)]* There is a r.v. B s.t. for all $\eta > 0$,
$$\int \mathbf{P}^z_\omega\big(\big\{\big|\tfrac{\sigma^2 X_n}{(\log n)^2} - B\big| > \eta\big\}\big)\, P(\mathrm{d}\omega) \underset{n\to\infty}{\longrightarrow} 0.$$
- *annealed functional CLT; [Seignourel (2000)]* For each $m\in\mathbb{N}$, consider an i.i.d. sequence $\omega(m) := (\omega_n(m))_{n\in\mathbb{N}}$ s.t. for all $z\in\mathbb{N}$,
$$\rho_z^+(m) := \rho_z^{1/\sqrt{m}},$$
and $X^{(m)}$ the RWRE w.r.t. $\omega(m)$. There is a **diffusion in RE** X s.t.
$$\big(\tfrac{1}{m}X^{(m)}_{\lfloor m^2 t\rfloor}\big)_{t\ge 0} \underset{m\to\infty}{\Longrightarrow} (X_t)_{t\ge 0}.$$

What do have these examples in common?

- Our (quenched) Markov processes on $\mathbb{R}$ are all **reversible** w.r.t. some Radon measure ν and

What do have these examples in common?

- Our (quenched) Markov processes on $\mathbb{R}$ are all **reversible** w.r.t. some Radon measure ν and **skip free**, i.e., they do not leave out points in $\mathrm{supp}(\nu)$.

What do have these examples in common?

- Our (quenched) Markov processes on $\mathbb{R}$ are all **reversible** w.r.t. some Radon measure ν and **skip free**, i.e., they do not leave out points in $\mathrm{supp}(\nu)$.
- MC in continuous time on discrete graphs are **uniquely determined** by their jump rates and transition probabilities.

What do have these examples in common?

- Our (quenched) Markov processes on $\mathbb{R}$ are all **reversible** w.r.t. some Radon measure ν and **skip free**, i.e., they do not leave out points in $\mathrm{supp}(\nu)$.
- MC in continuous time on discrete graphs are **uniquely determined** by their jump rates and transition probabilities.
- $\mathbb{R}$-valued symmetric Feller processes are uniquely determined by the scale metric r on $\mathbb{R}$ and the speed measure ν uniquely determined (up to a constant) by the **occupation time formula**: for all $x \in \mathrm{supp}(\nu)$, $R > 0$ and the MC X^R reflected at $\{-R, R\}$,

$$\mathbb{E}_x\Big[\int_0^{\tau_z} f(X_s^R)\mathrm{d}s\Big] = 2\int_{B(x,R)} r(z, c(x,y,z))f(y)\,\nu(\mathrm{d}y),$$

where $c(x,y,z)$ denotes the mid-point of three points $\{x,y,z\}$

What do have these examples in common?

- Our (quenched) Markov processes on $\mathbb{R}$ are all **reversible** w.r.t. some Radon measure ν and **skip free**, i.e., they do not leave out points in $\mathrm{supp}(\nu)$.
- MC in continuous time on discrete graphs are **uniquely determined** by their jump rates and transition probabilities.
- $\mathbb{R}$-valued symmetric Feller processes are uniquely determined by the scale metric r on $\mathbb{R}$ and the speed measure ν uniquely determined (up to a constant) by the **occupation time formula**: for all $x \in \mathrm{supp}(\nu)$, $R > 0$ and the MC X^R reflected at $\{-R, R\}$,

$$\mathbb{E}_x\Big[\int_0^{\tau_z} f(X_s^R)\mathrm{d}s\Big] = 2\int_{B(x,R)} r(z, c(x,y,z)) f(y)\,\nu(\mathrm{d}y),$$

where $c(x,y,z)$ denotes the mid-point of three points $\{x,y,z\}$

$\rightsquigarrow$ *BM with drift.* $r_c(x,y) = \frac{1}{2c}e^{-2c(x\wedge y)}(1 - e^{-2c|x-y|})$; $\nu(\mathrm{d}z) = e^{2cz}\mathrm{d}z$.

Stone's invariance principle

Theorem (Continuity in ν; [Stone (1963)])

Let $\nu, \nu_1, \nu_2, \ldots$ be Radon measures on $\mathbb{R}$, and $X, X_1, X_2, \ldots$ ν-symmetric Feller process resp. ν_n-symmetric Feller processes on $(\mathbb{R}, r_{eucl})$. If ν_n converges vaguely to ν and $\mathrm{supp}(\nu_n)$ converge in local Hausdorff topology to $\mathrm{supp}(\nu)$, then X_n converges weakly in path space to X.

$\rightsquigarrow$ Change of perspective: Identify our symmetric Feller processes with a **metric measure space**.

$\rightsquigarrow$ Generalize this in several directions:

1. provide invariance principle which relies on joint convergence of scale metric and speed measure,
2. formulate invariance principle for trees,
3. what to say beyond trees?

Motivation III: Symmetric RW on Kesten's tree

- Consider a GW-process (in discrete time) with critical offspring law with finite variance, and let (T, r, ϱ) be the rooted (random) family **tree conditioned on infinite height** with root ϱ.

Motivation III: Symmetric RW on Kesten's tree

- Consider a GW-process (in discrete time) with critical offspring law with finite variance, and let (T, r, ϱ) be the rooted (random) family **tree conditioned on infinite height** with root ϱ. Given a realization ω, let $X = ((X_n)_{n\geq 0}, \mathbf{P}^v_\omega, v \in T)$ the (discrete time) **symmetric RW**, i.e.,

$$p^\omega(v, v') := \tfrac{1}{\deg_\omega(v)} 1_{v\sim v'}.$$

Motivation III: Symmetric RW on Kesten's tree

- Consider a GW-process (in discrete time) with critical offspring law with finite variance, and let (T, r, ϱ) be the rooted (random) family **tree conditioned on infinite height** with root ϱ. Given a realization ω, let $X = ((X_n)_{n\geq 0}, \mathbf{P}^v_\omega, v \in T)$ the (discrete time) **symmetric RW**, i.e.,

$$p^\omega(v, v') := \tfrac{1}{\deg_\omega(v)} 1_{v \sim v'}.$$

- **annealed; [Kesten (1986)].** For $m \in \mathbb{N}$, put

$$\tau_m := \inf\{n : \; r(\varrho, X_n) = m\}.$$

For all $\epsilon > 0, m \in \mathbb{N}$, there are $x_1(\epsilon) > 0$, $x_2(\epsilon) < \infty$ s.t.,

$$\int \mathbf{P}^\varrho_\omega\big(\{x_1 \leq m^{-3}\tau_m \leq x_2\}\big) P_{\mathrm{GWtree}}(\mathrm{d}\omega) \geq 1 - \epsilon.$$

Motivation III: Symmetric RW on Kesten's tree

- Consider a GW-process (in discrete time) with critical offspring law with finite variance, and let (T, r, ϱ) be the rooted (random) family **tree conditioned on infinite height** with root ϱ. Given a realization ω, let $X = ((X_n)_{n\geq 0}, \mathbf{P}^v_\omega, v \in T)$ the (discrete time) **symmetric RW**, i.e.,

$$p^\omega(v, v') := \frac{1}{\deg_\omega(v)} 1_{v\sim v'}.$$

- **annealed; [Kesten (1986)].** For $m \in \mathbb{N}$, put

$$\tau_m := \inf\{n : \; r(\varrho, X_n) = m\}.$$

For all $\epsilon > 0, m \in \mathbb{N}$, there are $x_1(\epsilon) > 0$, $x_2(\epsilon) < \infty$ s.t.,

$$\int \mathbf{P}^\varrho_\omega\big(\{x_1 \leq m^{-3}\tau_m \leq x_2\}\big) P_{\mathrm{GWtree}}(\mathrm{d}\omega) \geq 1 - \epsilon.$$

Put $Z^n_t := n^{-\frac{1}{3}} \cdot r\big(\rho, X_{\lfloor nt\rfloor}\big)$, $t \geq 0$. Under $\int \mathbf{P}^\varrho_\omega \mathrm{d}P_{\mathrm{GWtree}}$ the family $\{Z^{(n)};\, n \in \mathbb{N}\}$ is tight.

Motivation III: Symmetric RW on Kesten's tree

- Consider a GW-process (in discrete time) with critical offspring law with finite variance, and let (T, r, ϱ) be the rooted (random) family **tree conditioned on infinite height** with root ϱ. Given a realization ω, let $X = ((X_n)_{n\geq 0}, \mathbf{P}^v_\omega, v \in T)$ the (discrete time) **symmetric RW**, i.e.,

$$p^\omega(v, v') := \frac{1}{\deg_\omega(v)} 1_{v\sim v'}.$$

- **annealed; [Kesten (1986)].** For $m \in \mathbb{N}$, put

$$\tau_m := \inf\{n : \; r(\varrho, X_n) = m\}.$$

For all $\epsilon > 0, m \in \mathbb{N}$, there are $x_1(\epsilon) > 0$, $x_2(\epsilon) < \infty$ s.t.,

$$\int \mathbf{P}^\varrho_\omega\big(\{x_1 \leq m^{-3}\tau_m \leq x_2\}\big) P_{\mathrm{GWtree}}(\mathrm{d}\omega) \geq 1 - \epsilon.$$

Put $Z^n_t := n^{-\frac{1}{3}} \cdot r\big(\rho, X_{\lfloor nt\rfloor}\big)$, $t \geq 0$. Under $\int \mathbf{P}^\varrho_\omega \mathrm{d}P_{\mathrm{GWtree}}$ the family $\{Z^{(n)};\, n \in \mathbb{N}\}$ is tight.

- **quenched; [Barlow & Kumagai (2006)].** For almost all realizations ω, under $\mathbf{P}^\varrho_\omega$ the family $\{Z^{(n)};\, n \in \mathbb{N}\}$ is NOT tight.

Overview

1. Set-up I: Trees and convergence in tree space.
   - metric trees and examples
   - Gromov-vague and Gromov-Hausdorff-vague convergence
2. Set-up II: Diffusions on continuum trees
   - definition and construction
   - examples
   - characterisation via the occupation time formula
3. The invariance principle
   - statement
   - characterizing tightness
   - extensions to non-tree spaces
   - back to our motivating examples

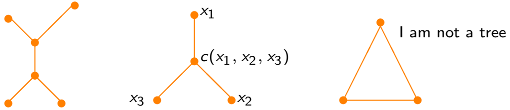
Set up I: Tree-like metric spaces

Definition

A **rooted metric tree** (T, r, ϱ) consists of a *distinguished point* $\varrho \in T$ and a *metric space* (T, r) which

1. satisfies the so-called **4-point condition** (equivalently, which is 0-hyperbolic), i.e., for all $x_1, x_2, x_3, x_4 \in X$,

$$r(x_1, x_2) + r(x_3, x_4) \leq \max\left\{r(x_1, x_3) + r(x_2, x_4), r(x_1, x_4) + r(x_2, x_3)\right\},$$



Definition

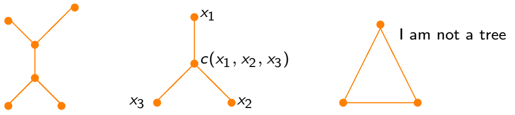
A **rooted metric tree** (T, r, ϱ) consists of a *distinguished point* $\varrho \in T$ and a *metric space* (T, r) which

1. satisfies the so-called **4-point condition** (equivalently, which is 0-hyperbolic), i.e., for all $x_1, x_2, x_3, x_4 \in X$,

$$r(x_1, x_2) + r(x_3, x_4) \leq \max\big\{r(x_1, x_3) + r(x_2, x_4), r(x_1, x_4) + r(x_2, x_3)\big\},$$

2. for any three points x_1, x_2, x_3 there exists a **branch point** $c(x_1, x_2, x_3) \in T$, i.e., such that

$$r(x_i, x_j) = r\big(x_i, c(x_1, x_2, x_3)\big) + r\big(c(x_1, x_2, x_3), x_j\big), \quad i \neq j \in \{1, 2, 3\}.$$



Set up I: Rooted $\mathbb{R}$-tree

Definition

A rooted metric tree (T, r, ϱ) is an **$\mathbb{R}$-tree** if it is *path-connected*, i.e., for all $x, y \in T$ there exists an isometry $\phi : [0, r(x, y)] \to T$ with $\phi_{x,y}(0) = x$ and $\phi_{x,y}(1) = y$.

Definition

A rooted metric tree (T, r, ϱ) is an $\mathbb{R}$-**tree** if it is *path-connected*, i.e., for all $x, y \in T$ there exists an isometry $\phi : [0, r(x, y)] \to T$ with $\phi_{x,y}(0) = x$ and $\phi_{x,y}(1) = y$.

Examples:

- $\mathbb{Z}$ or Kesten's tree with graph distance are metric trees but not $\mathbb{R}$-trees.
- $\mathbb{R}$ is a $\mathbb{R}$-tree.

Set up I: Rooted $\mathbb{R}$-tree

Definition

A rooted metric tree (T, r, ϱ) is an **$\mathbb{R}$-tree** if it is *path-connected*, i.e., for all $x, y \in T$ there exists an isometry $\phi : [0, r(x, y)] \to T$ with $\phi_{x,y}(0) = x$ and $\phi_{x,y}(1) = y$.

Examples:

- $\mathbb{Z}$ or Kesten's tree with graph distance are metric trees but not $\mathbb{R}$-trees.
- $\mathbb{R}$ is a $\mathbb{R}$-tree.

$\rightsquigarrow$ A metric tree (T, r) is a separable $\mathbb{R}$-tree if and only if it is the local Hausdorff limit of finite trees as edge lengths scale down to 0.

Obviously, $\mathbb{R}$ is the scaling limit of $\mathbb{Z}$.

Set up I: Metric measure tree

$\rightsquigarrow$ include the **speed measure** to allow for *time-change*

Definition

A **rooted metric measure tree** (T, r, ϱ, ν) consists of

- a rooted *Heine-Borel* metric tree (T, r, ρ), and
- a *boundedly finite* measure ν on $\mathcal{B}(T)$ of *full support*.

We call two rooted metric measure trees (T, r, ϱ, ν) and (T', r', ϱ', ν') equivalent if there is a isometry $\phi : T \to T'$ with $\phi(\varrho) = \varrho'$ and $\phi_* \nu = \nu'$.

$\mathbb{M} :=$ set of all equivalence classes of rooted metric measure trees.

$$\mathbb{M}_c := \big\{(T, r, \nu, \varrho) \in \mathbb{M} : (T, r) \text{ compact}\big\}.$$

Set up I: Rooted measure $\mathbb{R}$-trees and excursions

- Most prominent class of examples are rooted measure $\mathbb{R}$-trees "below" particular non-negative continuous functions:

$$\mathcal{E}_\infty := \left\{e : \mathbb{R} \to \mathbb{R}^+ : \; e(0) = 0, \; \lim_{x \to \pm\infty} \varphi(x) = \infty\right\}$$

$\varphi \in \mathcal{E}_\infty$ defines a pseudo-metric in $\mathbb{R}$, i.e., for all $x, y \in \mathbb{R}$,

$$r_\varphi\big(x, y\big) := \varphi(x) + \varphi(y) - 2 \inf_{s \in [x \wedge y, x \vee y]} \varphi(s).$$

Set up I: Rooted measure $\mathbb{R}$-trees and excursions

$\rightsquigarrow$ Most prominent class of examples are rooted measure $\mathbb{R}$-trees "below" particular non-negative continuous functions:

$$\mathcal{E}_\infty := \big\{ e : \mathbb{R} \to \mathbb{R}^+ : \; e(0) = 0, \; \lim_{x \to \pm\infty} \varphi(x) = \infty \big\}$$

$\varphi \in \mathcal{E}_\infty$ defines a pseudo-metric in $\mathbb{R}$, i.e., for all $x, y \in \mathbb{R}$,

$$r_\varphi\big(x, y\big) := \varphi(x) + \varphi(y) - 2 \inf_{s \in [x \wedge y, x \vee y]} \varphi(s).$$

Fact. After quotioning out, $T_\varphi := \mathbb{R}\big|_{\cong_\varphi}$ is a rooted Heine-Borel-$\mathbb{R}$-tree, i.e., a rooted $\mathbb{R}$-tree in which the closure of all balls are compact. In particular, T_φ is locally compact.

Set up I: Rooted measure $\mathbb{R}$-trees and excursions

$\rightsquigarrow$ Most prominent class of examples are rooted measure $\mathbb{R}$-trees "below" particular non-negative continuous functions:

$$\mathcal{E}_\infty := \left\{e : \mathbb{R} \to \mathbb{R}^+ :\ e(0) = 0,\ \lim_{x\to\pm\infty} \varphi(x) = \infty\right\}$$

$\varphi \in \mathcal{E}_\infty$ defines a pseudo-metric in $\mathbb{R}$, i.e., for all $x, y \in \mathbb{R}$,

$$r_\varphi(x, y) := \varphi(x) + \varphi(y) - 2 \inf_{s\in[x\wedge y, x\vee y]} \varphi(s).$$

Fact. After quotioning out, $T_\varphi := \mathbb{R}\big|_{\cong_\varphi}$ is a rooted Heine-Borel-$\mathbb{R}$-tree, i.e., a rooted $\mathbb{R}$-tree in which the closure of all balls are compact. In particular, T_φ is locally compact.

$\rightsquigarrow$ Let in addition ν_φ be the push forward of the Lebesgue measure on $\mathbb{R}$ under the map sends $t \in \mathbb{R}$ to a point in the tree $(\mathbb{R}\big|_{\cong_\varphi}, r_\varphi)$.

Example. If

$$\varphi(t) := Y^1_t 1_{[0,\infty)}(t) + Y^2_{-t} 1_{(-\infty,0]}(t),$$

where Y^1 and Y^2 are two independent copies of the solution of

$$\mathrm{d}Y_t = \tfrac{1}{Y_t}\mathrm{d}t + \mathrm{d}B_t, \qquad Y_0 = 0,$$

Gromov-vague and Gromov-Hausdorff-vague convergence

Let $\mathcal{X}_n := (T_n, r_n, \nu_n, \varrho_n)_{n\in\mathbb{N}}$, $\mathcal{X} := (T, r, \nu, \varrho)$ be in $\mathbb{M}_c$.

$\rightsquigarrow$ In which sense shall our rooted metric measure trees converge?

We say that

- *Gromov-weak.*
- *Gromov-Hausdorff-weak.*
- *Gromov-(Hausdorff)-vague.*

Gromov-vague and Gromov-Hausdorff-vague convergence

Let $\mathcal{X}_n := (T_n, r_n, \nu_n, \varrho_n)_{n\in\mathbb{N}}$, $\mathcal{X} := (T, r, \nu, \varrho)$ be in $\mathbb{M}_c$.

$\rightsquigarrow$ In which sense shall our rooted metric measure trees converge?

We say that

- *Gromov-weak.* $\mathcal{X}_n \underset{n\to\infty}{\longrightarrow} \mathcal{X}$ **Gromov-weakly**, iff there is a pointed *compact* metric space (Z, r_Z, ϱ_Z) and isometric embeddings $\phi_n : T_n \to Z$ and $\phi : T \to Z$ with
    - $\phi_n(\varrho) = \phi(\varrho) = \varrho_Z$, and
    - $(\phi_n)_*\mu_n \underset{n\to\infty}{\Longrightarrow} \phi_*\mu$.
- *Gromov-Hausdorff-weak.*
- *Gromov-(Hausdorff)-vague.*

Gromov-vague and Gromov-Hausdorff-vague convergence

Let $\mathcal{X}_n := (T_n, r_n, \nu_n, \varrho_n)_{n\in\mathbb{N}}$, $\mathcal{X} := (T, r, \nu, \varrho)$ be in $\mathbb{M}_c$.

$\rightsquigarrow$ In which sense shall our rooted metric measure trees converge?

We say that

- *Gromov-weak.*

  Example. Let $\mathcal{X}_n := \big(\{1,2\}, r_n(1,2) = 1, \frac{1}{n}\delta_1 + \big(1 - \frac{1}{n}, \varrho_n := 1\big)\delta_2\big)$

  Obviously, $\mathcal{X}_n \underset{n\to\infty}{\longrightarrow} \mathcal{X} := \overline{\big(\{1,2\}, \rho = 1, \delta_2\big)}$ Gromov-weakly. However, the **supports do not converge**.

- *Gromov-Hausdorff-weak.*
- *Gromov-(Hausdorff)-vague.*

Gromov-vague and Gromov-Hausdorff-vague convergence

Let $\mathcal{X}_n := (T_n, r_n, \nu_n, \varrho_n)_{n\in\mathbb{N}}$, $\mathcal{X} := (T, r, \nu, \varrho)$ be in $\mathbb{M}_c$.

$\rightsquigarrow$ In which sense shall our rooted metric measure trees converge?

We say that

- *Gromov-weak.*
- *Gromov-Hausdorff-weak.* $\mathcal{X}_n \underset{n\to\infty}{\longrightarrow} \mathcal{X}$ **Gromov-Hausdorff-weakly**, iff there is a pointed *compact* metric space (Z, r_Z, ϱ_Z) and isometric embeddings $\phi_n : T_n \to Z$ and $\phi : T \to Z$ with
  - $\phi_n(\varrho) = \phi(\varrho) = \varrho_Z$, and
  - $(\phi_n)_*\mu_n \underset{n\to\infty}{\Longrightarrow} \phi_*\mu$, and in addition
  - T_n **converges to** $\mathrm{supp}(\mu)$ **in Hausdorff topology**.
- *Gromov-(Hausdorff)-vague.*

Gromov-vague and Gromov-Hausdorff-vague convergence

Let $\mathcal{X}_n := (T_n, r_n, \nu_n, \varrho_n)_{n\in\mathbb{N}}$, $\mathcal{X} := (T, r, \nu, \varrho)$ be in $\mathbb{M}_c$.

$\rightsquigarrow$ In which sense shall our rooted metric measure trees converge?

We say that

- *Gromov-weak.*
- *Gromov-Hausdorff-weak.*
- *Gromov-(Hausdorff)-vague.* We extend this to $\mathbb{M}$ by localization, i.e., we say that $\mathcal{X}_n$ converges to $\mathcal{X}$ **Gromov-(Hausdorff)-vaguely** iff for almost all $R > 0$,

$$(T_n, r_n, (\nu_n)|_{\bar{B}_n(\varrho_n,R)}, \varrho_n) \underset{n\to\infty}{\longrightarrow} (T, r, (\nu)|_{\bar{B}(\varrho,R)}, \varrho),$$

Gromov-(Hausdorff)-weakly.

The speed ν-motion on trees

Let (T, r, ϱ) be a rooted Heine-Borel metric tree. We can define:

- a **length measure** by $\lambda^{(T,r,\varrho)}((\varrho, x]) = r(\varrho, x)$, $x \in T$, and
- for all absolutely continuous functions $f \in \mathcal{A}$ a **gradient** ∇f s.t.

$$f(y) - f(x) = \int_x^y \nabla f \mathrm{d}\lambda, \qquad \forall x, y \in T.$$

Theorem ([Athreya, Eckhoff & W. (2013)], [Athreya, Löhr & W. (2017)])

Let (T, r, ν) be a metric measure space. There exists a unique (up to ν-equivalence) ν-symmetric Markov process $X^{(T,r,\nu)} := (X_t)_{t \geq 0}$ whose Dirichlet form is given by the closure of

$$\mathcal{E}(f, g) := \tfrac{1}{2} \int \nabla f \nabla g \mathrm{d}\lambda^{(T,r,\rho)},$$
$$\mathcal{D}(\mathcal{E}) := \{f \in \mathcal{A} \cap L^2(\nu) \cap \mathcal{C}_\infty : \mathcal{E}(f, f) < \infty\}.$$

$\rightsquigarrow$ No need for closing the form if $\inf_{x \in T} \nu\big(B(x, \delta)\big) > 0$ for all $\delta > 0$.

The speed ν-motion on trees: Examples

Let $X^{(T,r,\nu)}$ be the speed-ν motion on (T,r).

- If $(T,r,\nu) = (\mathbb{R}, r_{\mathrm{eucl}}, \frac{1}{\sigma^2(x)}\mathrm{d}x)$, then

$$\mathrm{d}X_t^{(T,r,\nu)} = \sigma(X_t^{(T,r,\nu)})\mathrm{d}B_t.$$

- If (T,r,ν) is a *finite* metric measure tree, then $X^{(T,r,\nu)}$ jumps from x to $y \sim x$ at rate

$$\gamma_{x,y} := \frac{1}{2\nu(\{x\})r(x,y)}.$$

Now assume *unit edge lengths*.

- If ν is the counting measure, then we have *degree-dependent* total jump rates

$$\gamma_x := \sum_{y\sim x} \gamma_{x,y} = \tfrac{1}{2}\mathrm{deg}(x).$$

- If $\nu := \sum_{x\in T} \delta_{\mathrm{deg}(x)}$, then we have *constant* total jump rates.

The speed ν-motion on trees: A useful properties

1. $X^{(T,r,\nu)}$ takes values in

$$\mathcal{C}l_T([0,\infty)) := \{\omega \in \mathcal{D}_T([0,\infty)) : \omega([a,b]) \text{ closed } \forall\, a < b\}.$$

The speed ν-motion on trees: A useful properties

1. $X^{(T,r,\nu)}$ takes values in

$$\mathcal{C}l_T([0,\infty)) := \big\{\omega \in \mathcal{D}_T([0,\infty)) : \omega([a,b]) \text{ closed } \forall\, a < b\big\}.$$

2. $X^{(T,r,\nu)}$ has continuous sample paths iff (T,r) is an $\mathbb{R}$-tree.

The speed ν-motion on trees: A useful properties

1. $X^{(T,r,\nu)}$ takes values in

$$\mathcal{C}I_T([0,\infty)) := \{\omega \in \mathcal{D}_T([0,\infty)) : \omega([a,b]) \text{ closed } \forall\, a < b\}.$$

2. $X^{(T,r,\nu)}$ has continuous sample paths iff (T,r) is an $\mathbb{R}$-tree.

Proposition ([Athreya, Eckhoff & W. (2013)])

Let (T,r) be compact.

1. *The speed-ν motion $X^{(T,r,\nu)}$ on (T,r) satisfies the occupation time formula: for $x,z \in T$,*

$$\mathbb{E}_x\Big[\int_0^{\tau_z} f(X_s)\,\mathrm{d}s\Big] = 2\int_T f(y) r\big(z, c(x,y,z)\big)\,\nu(\mathrm{d}y).$$

2. *If X is a strong Markov process which satisfies the above occupation time formula, then X is the speed-ν motion on (T,r).*

How to deal with different state spaces?

Let $X, X^1, X^2, \ldots$ be càdlàg processes with values in a metric space $(T, r), (T_1, r_1), (T_2, r_2), \ldots$ We say that

$$(X^n)_{n\in\mathbb{N}} \text{ converges to } X$$

weakly in path space (resp. f.d.d.) if there exist a metric space (Z, d_Z) and isometric embeddings $\phi_n\colon T_n \to Z$, $n \in \mathbb{N}$ and $\phi : T \to Z$, such that

$$(\phi_n \circ X^n)_{n\in\mathbb{N}} \text{ converges to } \phi_\infty \circ X$$

weakly in path space (resp. f.d.d.).

The invariance principle

For A, B open, define **resistance**,

$$R(A,B) := \left(\inf\left\{\mathcal{E}(f,f):\ f\in\mathcal{D}(\mathcal{E}), f|_A = 1, f|_B = 0\right\}\right)^{-1}$$

and let for $x \in T$, $R(x,B) := \sup_{A\ni x} R(A,B)$.

Theorem (Athreya, Löhr & W. (2017))

Let $\mathcal{X}_n = (T_n, r_n, \nu_n, \varrho_n)_{n\in\mathbb{N}}$ and $\mathcal{X} = (T, r, \nu, \varrho)$ be in $\mathbb{M}$. Assume that

$$\lim_{R\to\infty}\liminf_{n\to\infty} R_n(\varrho_n, \complement\bar{B}(\varrho_n, R)) = \infty. \qquad (*)$$

Then

$$\mathcal{X}_n \overset{GHvag}{\longrightarrow} \mathcal{X} \ \Rightarrow\ \mathcal{L}_{\rho_n}(X^{\mathcal{X}_n}) \underset{n\to\infty}{\Longrightarrow} \mathcal{L}_{\rho}(X^{\mathcal{X}}).$$

$\leadsto$ Condition $(*)$ ensures that limit points have *infinite life time*.

Earlier version of this invariance principles.

- For RWs on graph-trees to diffusions on $\mathbb{R}$-trees restricted to homogeneous rescaling ([Croydon (2010)])

The gap between Gromov-vague and Gromov-Hausdorff-vague

Write $\mathcal{X} := (T, r, \nu, \varrho)$, and put for $\delta > 0$,

$$m_\delta(\mathcal{X}) := \inf_{x \in T} \nu\big(B_r(x, \delta)\big).$$

As by assumption $\mathrm{supp}(\nu) = T$, we have $m_\delta(\mathcal{X}) > 0$.

1. **Gromov-Hausdorff-weak convergence** is equivalent to
   (a) **Gromov-weak convergence**, and
   (b) the **Uniform lower mass bound property (ULMB)**, i.e., for all $\delta > 0$, $\inf_{n \in \mathbb{N}} m_\delta(\mathcal{X}_n) > 0$.

([Athreya, Löhr & W. (2016)])

The gap between Gromov-vague and Gromov-Hausdorff-vague

Write $\mathcal{X} := (T, r, \nu, \varrho)$, and put for $\delta > 0$,

$$m_\delta(\mathcal{X}) := \inf_{x \in T} \nu\big(B_r(x, \delta)\big).$$

As by assumption $\mathrm{supp}(\nu) = T$, we have $m_\delta(\mathcal{X}) > 0$.

1. **Gromov-Hausdorff-weak convergence** is equivalent to
   (a) **Gromov-weak convergence**, and
   (b) the **Uniform lower mass bound property (ULMB)**, i.e., for all $\delta > 0$, $\inf_{n \in \mathbb{N}} m_\delta(\mathcal{X}_n) > 0$.

   ([Athreya, Löhr & W. (2016)])

2. If *(ULMB)* fails, we still have $X^{(T_n, r_n, \nu_n)} \xrightarrow[n \to \infty]{\text{f.d.d.}} X^{(T, r, \nu)}$.
   ([Athreya, Löhr & W. (2017)])

   That is, the ULMB is the gap between f.d.d.convergence and weak convergence in path space.

Beyond an invariance principles for trees

- *rate of convergence.* [Rojas-Barragan, im preparation] equips $\mathcal{M}_1(\mathcal{C}I([0,\infty)))$ with a complete metric d inducing the weak topology w.r.t. the Skorohod topology, and $\mathbb{M}$ with a complete metric d_{suppGHPr} such that

$$d\big(\mathcal{L}(X^{\mathcal{X}_1}), \mathcal{L}(X^{\mathcal{X}_2})\big) \cong d_{\text{suppGHPr}}\big(\mathcal{X}_1, \mathcal{X}_2\big).$$

- *resistance networks.* [Croydon (2018)] generalizes invariance principle to resistance networks (graph-like metric spaces, fractals) by replacing the metric on the tree by the resistance metric

$$R(x,y) := \cap_{A\ni x, B\ni y} R(A,B).$$

  Crucial assumptions: $R(x,y) < \infty$ for all $x, y \in T$.

- *general invariance principle.* [Gerle, im preparation] uses the resistance $R(A,B)$ to construct a uniformity. He then states that the stochastic processes converge if the associated uniform spaces converge.

Example I: Symmetric RW on $\mathbb{Z}$

For each $n \in \mathbb{N}$, put

$$T_n := \mathbb{Z}, \quad r_n\big(v, v \pm 1\big) := \tfrac{1}{n}, \quad \nu_n(\{v\}) := \tfrac{1}{n}, \quad \forall v \in \mathbb{Z}.$$

The *speed-ν_n random walk on (T_n, r_n)* is the symmetric RW on $\mathbb{Z}$ with edge length re-scaled by $\frac{1}{n}$ and speeded up (in each vertex v) by a factor of

$$\gamma_n(v) := \tfrac{1}{2\nu_n(\{v\})} \sum_{v'=v\pm 1} r_n^{-1}(v, v') = \tfrac{1}{2} \cdot n \cdot 2n = n^2.$$

Then

- $(T_n, r_n, \nu_n, 0) \underset{n\to\infty}{\longrightarrow} (\mathbb{R}, r_{\mathrm{eucl}}, \mathrm{d}x, 0)$ Gromov-Hausdorff-vaguely.
- $X^{(\mathbb{R}, r_{\mathrm{eucl}}, \mathrm{d}x, 0)}$ is standard BM.
- $k \geq R_n(0, \{-k, k\}) \geq k/2$, $k \in \mathbb{N}$, on $\mathbb{Z}$.

$\rightsquigarrow$ *Classical functional CLT.* $\mathcal{L}_0\big(X^{(T_n, r_n, \nu_n)}\big) \underset{n\to\infty}{\Longrightarrow} \mathcal{L}_0\big(X^{(\mathbb{R}, r_{\mathrm{eucl}}, \mathrm{d}x)}\big)$.

Example II: Sinai's RWRE on $\mathbb{Z}$ (quenched)

Fix $\omega^- \in (0,1)^{\mathbb{Z}}$. Put $\omega_x^+ = 1 - \omega_x^-$ and $\rho_x := \frac{\omega_x^-}{\omega_x^+}$. Let $X^\omega = ((X_t^\omega)_{t\geq 0}, \mathbf{P}_z^\omega, z \in \mathbb{Z})$ be the MC on $\mathbb{Z}$ with jump rates

$$\gamma^\omega_{z,z\pm 1} := \omega_z^\pm$$

1.
2.

Example II: Sinai's RWRE on $\mathbb{Z}$ (quenched)

Fix $\omega^- \in (0,1)^{\mathbb{Z}}$. Put $\omega_x^+ = 1 - \omega_x^-$ and $\rho_x := \frac{\omega_x^-}{\omega_x^+}$. Let $X^\omega = ((X_t^\omega)_{t\geq 0}, \mathbf{P}_z^\omega, z \in \mathbb{Z})$ be the MC on $\mathbb{Z}$ with jump rates

$$\gamma_{z,z\pm 1}^\omega := \omega_z^\pm$$

1. X is reversible w.r.t. the unique (up to a constant) measure ν with $\nu_\omega(\{x+1\})\omega_{x+1}^- = \nu_\omega(\{x\})\omega_x^+$, and therefore

$$\nu_\omega(\{x\}) := C(1+\rho_x)e^{-\mathrm{sign}(x)\sum_{k=0}^{|x|-1}\log(\rho_{k+x\wedge 0})}.$$

   If we choose r_ω such that for all $y < z$,

$$r_\omega(y,z) = \frac{1}{2}\sum_{x=y}^{z-1}\frac{1+\rho_x}{\nu_\omega(\{x\})} = \frac{1}{2C}\sum_{x=y}^{z-1} e^{\mathrm{sign}(x)\sum_{k=0}^{|x|-1}\log(\rho_{k+x\wedge 0})},$$

   then $X^{(\mathbb{Z}, r_\omega, \nu_\omega)}$ is the speed-ν_ω motion on $(\mathbb{Z}, r_\omega)$.

2.

Example II: Sinai's RWRE on $\mathbb{Z}$ (quenched)

Fix $\omega^- \in (0,1)^{\mathbb{Z}}$. Put $\omega_x^+ = 1 - \omega_x^-$ and $\rho_x := \frac{\omega_x^-}{\omega_x^+}$. Let $X^\omega = ((X_t^\omega)_{t\geq 0}, \mathbf{P}_z^\omega, z \in \mathbb{Z})$ be the MC on $\mathbb{Z}$ with jump rates

$$\gamma^\omega_{z,z\pm 1} := \omega_z^\pm$$

1. If we now let $\rho_x(m) := \rho_x^{m^{-1/2}}$, then we obtain

$$\nu_{\omega(m)}(\{x\}) := C(1+\rho_x^{\frac{1}{\sqrt{m}}})e^{-\mathrm{sign}(x)\frac{1}{\sqrt{m}}\sum_{k=0}^{|x|-1}\log(\rho_{k+x\wedge m})}.$$

$$r_{\omega(m)}(y,z) = \frac{1}{2C}\sum_{x=y}^{z-1} e^{\mathrm{sign}(x)\frac{1}{\sqrt{m}}\sum_{k=0}^{|x|-1}\log(\rho_{k+x\wedge 0})}.$$

2.

Example II: Sinai's RWRE on $\mathbb{Z}$ (quenched)

Fix $\omega^- \in (0,1)^{\mathbb{Z}}$. Put $\omega_x^+ = 1 - \omega_x^-$ and $\rho_x := \frac{\omega_x^-}{\omega_x^+}$. Let $X^\omega = ((X_t^\omega)_{t\geq 0}, \mathbf{P}_z^\omega, z \in \mathbb{Z})$ be the MC on $\mathbb{Z}$ with jump rates

$$\gamma^\omega_{z,z\pm 1} := \omega_z^\pm$$

1. If we now let $\rho_x(m) := \rho_x^{m^{-1/2}}$, then we obtain

$$\nu_{\omega(m)}(\{x\}) := C(1+\rho_x^{\frac{1}{\sqrt{m}}})e^{-\mathrm{sign}(x)\frac{1}{\sqrt{m}}\sum_{k=0}^{|x|-1}\log(\rho_{k+x\wedge m})}.$$

$$r_{\omega(m)}(y,z) = \frac{1}{2C}\sum_{x=y}^{z-1} e^{\mathrm{sign}(x)\frac{1}{\sqrt{m}}\sum_{k=0}^{|x|-1}\log(\rho_{k+x\wedge 0})}.$$

2. Hence, **if** $\frac{1}{\sqrt{m}}\sum_{k=0}^{|mx|-1}\log(\rho_x) \underset{m\to\infty}{\longrightarrow} w(x)$ for all $x \in \mathbb{Z}$, then

$$\frac{1}{m}r_{\omega(m)}(y,z) \underset{m\to\infty}{\longrightarrow} d_\omega(y,z) := \int_y^z e^{w(x)}\mathrm{d}x,\ \frac{1}{m}\nu_{\omega(m)} \overset{\text{vag}}{\longrightarrow} \mu_\omega := 2\int_\cdot e^{-w(x)}\mathrm{d}x.$$

That implies that

$$\mathcal{L}_0\big(\tfrac{1}{m}X^{\omega(m)}_{m^2\cdot}\big) \underset{m\to\infty}{\Longrightarrow} \mathcal{L}_0\big(X_\cdot^{(\mathrm{R},d_\omega,\mu_\omega)}\big).$$

Example II: Sinai's RWRE on $\mathbb{Z}$ (annealed)

$$V_{\omega(m)}(x) := \frac{1}{\sqrt{m}} \sum_{k=0}^{|mx|-1} \log\left(\rho_{k+x\wedge 0}\right)$$

Theorem ([Andriopoulos (arXiv:1812.10197)])

Assume that $\{\omega_x^-;\, x \in \mathbb{Z}\}$ are random and are distributed such that

$$\left(V_{\omega(m)}(\lfloor xm \rfloor)\right)_{x\in\mathbb{R}} \underset{m\to\infty}{\Longrightarrow} \left(W(x)\right)_{x\in\mathbb{R}},$$

where W is two-sided BM, then

$$\left(\tfrac{1}{m} X^{(m)}_{m^2 t}\right)_{t\geq 0} \underset{m\to\infty}{\longrightarrow} (X_t)_{t\geq 0},$$

where X can be identified as Brox-diffusion (BM in a BM medium).

$\rightsquigarrow$ This way, [Andriopoulos (arXiv:1812.10197)] could relax the i.i.d. assumption on the medium and does not assume *uniform ellipticity*.

Example III: Kesten's tree in the continuum limit

Let $\mathcal{T}_{\mathrm{Kesten}}$ be the GW-process (in discrete time) with critical offspring law with finite variance conditioned to stay alive for ever. Given a realization ω, let $X = ((X_n)_{n\geq 0}, \mathbf{P}^v_\omega, v \in T)$ the (discrete time) **symmetric RW**, i.e.,

$$p^\omega(v, v') := \frac{1}{\deg_\omega(v)} 1_{v\sim v'}.$$

Example III: Kesten's tree in the continuum limit

Let $\mathcal{T}_{\text{Kesten}}$ be the GW-process (in discrete time) with critical offspring law with finite variance conditioned to stay alive for ever. Given a realization ω, let $X = ((X_n)_{n\geq 0}, \mathbf{P}^v_\omega, v \in T)$ the (discrete time) **symmetric RW**, i.e.,

$$p^\omega(v,v') := \frac{1}{\deg_\omega(v)} 1_{v\sim v'}.$$

1. Let $S = (S_n)_{n\in\mathbb{N}}$ be a symmetric RW on $\mathbb{Z}$ and B BM on $\mathbb{R}$, and put

   $$\hat{S}_n := S_n - 2 \inf_{0\leq m\leq n} S_m \text{ and } \hat{B}_t := B_t - 2 \inf_{0\leq s\leq t} B_s.$$

   Write $\bar{S}$ and $\bar{B}$ for the *two-sided* versions. Clearly,

   $$\left(\tfrac{1}{m}\bar{S}_{tm^2}\right)_{t\in\mathbb{R}} \underset{m\to\infty}{\Longrightarrow} \left(\bar{B}_t\right)_{t\in\mathbb{R}}.$$

Example III: Kesten's tree in the continuum limit

Let $\mathcal{T}_{\mathrm{Kesten}}$ be the GW-process (in discrete time) with critical offspring law with finite variance conditioned to stay alive for ever. Given a realization ω, let $X = ((X_n)_{n\geq 0}, \mathbf{P}^v_\omega, v \in T)$ the (discrete time) **symmetric RW**, i.e.,

$$p^\omega(v,v') := \tfrac{1}{\deg_\omega(v)} 1_{v\sim v'}.$$

1. Let $S = (S_n)_{n\in\mathbb{N}}$ be a symmetric RW on $\mathbb{Z}$ and B BM on $\mathbb{R}$, and put
$$\hat{S}_n := S_n - 2 \inf_{0\leq m\leq n} S_m \text{ and } \hat{B}_t := B_t - 2 \inf_{0\leq s\leq t} B_s.$$
Write $\bar{S}$ and $\bar{B}$ for the *two-sided* versions. Clearly,
$$\big(\tfrac{1}{m}\bar{S}_{tm^2}\big)_{t\in\mathbb{R}} \underset{m\to\infty}{\Longrightarrow} \big(\bar{B}_t\big)_{t\in\mathbb{R}}.$$

2. If the offspring is *geometric*, then Kesten's tree is the rooted mm-tree $(T_{\bar{S}}, r_{\bar{S}}, \nu_{\bar{S}}, 0)$ where $(T_{\bar{S}}, r_{\bar{S}})$ is associated with $\bar{S}$ and $\nu_{\bar{S}}$ is the push forward of the counting measure on $\mathbb{Z}$ into the tree.

Example III: Kesten's tree in the continuum limit

Let $\mathcal{T}_{\text{Kesten}}$ be the GW-process (in discrete time) with critical offspring law with finite variance conditioned to stay alive for ever. Given a realization ω, let $X = ((X_n)_{n\geq 0}, \mathbf{P}^v_\omega, v \in T)$ the (discrete time) **symmetric RW**, i.e.,

$$p^\omega(v, v') := \frac{1}{\deg_\omega(v)} 1_{v\sim v'}.$$

1. Let $S = (S_n)_{n\in\mathbb{N}}$ be a symmetric RW on $\mathbb{Z}$ and B BM on $\mathbb{R}$, and put
$$\hat{S}_n := S_n - 2 \inf_{0\leq m\leq n} S_m \text{ and } \hat{B}_t := B_t - 2 \inf_{0\leq s\leq t} B_s.$$
Write $\bar{S}$ and $\bar{B}$ for the *two-sided* versions. Clearly,
$$\left(\tfrac{1}{m}\bar{S}_{tm^2}\right)_{t\in\mathbb{R}} \underset{m\to\infty}{\Longrightarrow} \left(\bar{B}_t\right)_{t\in\mathbb{R}}.$$

2. If the offspring is *geometric*, then Kesten's tree is the rooted mm-tree $(T_{\bar{S}}, r_{\bar{S}}, \nu_{\bar{S}}, 0)$ where $(T_{\bar{S}}, r_{\bar{S}})$ is associated with $\bar{S}$ and $\nu_{\bar{S}}$ is the push forward of the counting measure on $\mathbb{Z}$ into the tree.

$\rightsquigarrow$ [Pitman (1975)]. $\hat{B}$ is the unique solution of the SDE

$$\mathrm{d}Y_t := \frac{1}{Y_t}\mathrm{d}t + B_t.$$

Example III: Scaling Kesten's tree

For each $m \in \mathbb{N}$, let $\mathcal{T}_m$ **be the Kesten's-tree conditioned to never die out** and put for $v \sim v'$,

$$T_m := \mathbb{R}|_{\cong \hat{S}} \quad r_m(v, v') := \tfrac{1}{m} r|_{\cong \hat{S}}(v, v'), \quad \nu_m(\{v\}) := \tfrac{\deg(v)}{2m^2}.$$

The **speed-ν_m-speed RW on** (T_m, r_m) is the symmetric RW on $\mathbb{Z}$ with edge length re-scaled by $\frac{1}{m}$ and speeded up (in each vertex v) by a factor of

$$\gamma_m(v) = \tfrac{1}{2\nu_m(\{v\})} \sum_{v' \sim v} r_m^{-1}(v, v') = \tfrac{1}{2} \cdot \tfrac{2m^2}{\deg(v)} \cdot \deg(v) m = m^3.$$

Example III: Scaling Kesten's tree

For each $m \in \mathbb{N}$, let $\mathcal{T}_m$ **be the Kesten's-tree conditioned to never die out** and put for $v \sim v'$,

$$T_m := \mathbb{R}|_{\cong \hat{S}} \quad r_m(v, v') := \tfrac{1}{m} r|_{\cong \hat{S}}(v, v'), \quad \nu_m(\{v\}) := \tfrac{\deg(v)}{2m^2}.$$

The **speed-ν_m-speed RW on (T_m, r_m)** is the symmetric RW on $\mathbb{Z}$ with edge length re-scaled by $\frac{1}{m}$ and speeded up (in each vertex v) by a factor of

$$\gamma_m(v) = \frac{1}{2\nu_m(\{v\})} \sum_{v' \sim v} r_m^{-1}(v, v') = \frac{1}{2} \cdot \frac{2m^2}{\deg(v)} \cdot \deg(v) m = m^3.$$

Theorem ([Athreya, Löhr & W. (2017)])

The sequence $(T_m, r_m, \nu_m, 0)$ converges weakly in Gromov-Hausdorff-vague topology to the tree $(T_Y, r_Y, \nu_Y, 0)$ associated with the two-sided Y.

Example III: Symmetric RW on Kesten's tree

- *annealed.* Under the annealed law, if X is the symmetric RW on Kesten's tree, then

$$\mathcal{L}_0\big((\tfrac{1}{m}X_{tm^3})_{t\geq 0}\big) \underset{m\to\infty}{\Longrightarrow} \mathcal{L}_0\big(X^{(T_Y, r_Y, \nu_Y)}\big).$$

Example III: Symmetric RW on Kesten's tree

- *annealed.* Under the annealed law, if X is the symmetric RW on Kesten's tree, then

$$\mathcal{L}_0\Big(\big(\tfrac{1}{m}X_{tm^3}\big)_{t\geq 0}\Big) \underset{m\to\infty}{\Longrightarrow} \mathcal{L}_0\big(X^{(T_Y, r_Y, \nu_Y)}\big).$$

Put $Z_n := r_Y(0, X_n)$. By continuity,

$$\big(\frac{1}{m}Z_{m^3 t}\big)_{t\geq 0} \underset{m\to\infty}{\Longrightarrow} \big(r_Y(0, X^{(T_Y, r_Y, \nu_Y)})\big)_{t\geq 0}.$$

Example III: Symmetric RW on Kesten's tree

- *annealed.* Under the annealed law, if X is the symmetric RW on Kesten's tree, then

$$\mathcal{L}_0\big((\tfrac{1}{m}X_{tm^3})_{t\geq 0}\big) \underset{m\to\infty}{\Longrightarrow} \mathcal{L}_0\big(X^{(T_Y,r_Y,\nu_Y)}\big).$$

Put $Z_n := r_Y(0,X_n)$. By continuity,

$$(\frac{1}{m}Z_{m^3t})_{t\geq 0} \underset{m\to\infty}{\Longrightarrow} (r_Y(0,X^{(T_Y,r_Y,\nu_Y)}))_{t\geq 0}.$$

As $X^{(T_Y,r_Y,\nu_Y)}$ is recurrent, the limit distance process is recurrent (and therefore non-trivial).

- *quenched.* For almost all realizations of a two-sided BM,

$$\liminf_{n\to\infty} \nu_n\big(B_n(0,R)\big) = 0, \quad \text{and} \quad \limsup_{n\to\infty} \nu_n\big(B_n(0,R)\big) = \infty$$

([Barlow & Kumagai (2006)]). Thus the sequence $\{\nu_n;\, n\in\mathbb{N}\}$ does not converge. Moreover, under the suggested re-scaling the RWs are NOT tight.

This is the end

Obrigada

Literature

George Andriopoulos. Invariance principles for random walks in random environment on trees, arxiv:1812.10197.

Siva Athreya, Michael Eckhoff and Anita Winter (2013). Brownian motion on real trees, Trans. AMS.

Siva Athreya, Wolfgang Löhr and Anita Winter (2017). Invariance principle for variable speed random walks on trees, AoP.

Siva Athreya, Wolfgang Löhr and Anita Winter (2016). The gap between Gromov-Hausdorff-vague and Gromov-vague topology, SPA.

Martin Barlow and Takashi Kumagai (2006). Random walk on the incipient infinite cluster on trees, Illinois Journal of Mathematics

David Croydon (2010), Scaling limits for simple random walks on random ordered graph trees, Adv. Appl Probab

David Croydon, Scaling limits of stochastic processes associated with resistance networks, arXiv.1609.05666

Harry Kesten (1986) Subdiffusive behaviour of a random walk on a random cluster, Poincare

Jun Kigami (2012) Resistance forms, quasisymmetric maps and heat kernel estimates, Mem. AMS

James Pitman (1975), One-Dimensional Brownian Motion and the Three-Dimensional Bessel Process, Advances in Appl. Probab.

Paul Seignourel, Discrete schemes for processes in random media, PTRF.

Charles Stone (1963), Limit theorems for random walks, birth and death processes, and diffusion processes, Illinois Journal of Mathematics