Stability, convergence to equilibrium and simulation of non-linear Hawkes Processes with memory kernels given by the sum of Erlang kernels

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joint work with E. Löcherbah and G. Ost

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Hawkes process

Let N be a counting process on $\mathbb{R}_+$ characterised by its intensity process $(\lambda_t)_{t\geq 0}$ defined, for each $t \geq 0$, through the relation

$$\mathbb{P}(N \text{ has a jump in }]t, t+dt]|\mathcal{F}_t) = \lambda_t dt,$$

where $\mathcal{F}_t = \sigma(N(]u,s]), 0 \leq u < s \leq t)$

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where $\mathcal{F}_t = \sigma(N(]u,s]), 0 \leq u < s \leq t)$ and

$$\lambda_t = f\left(\delta + \int_{]0,t[} h(t-s) dN_s\right). \tag{1}$$

Here, $f : \mathbb{R} \to \mathbb{R}_+$ is the *jump rate function* and $h : \mathbb{R}_+ \to \mathbb{R}$ is the *memory kernel.*

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Here, $f : \mathbb{R} \to \mathbb{R}_+$ is the *jump rate function* and $h : \mathbb{R}_+ \to \mathbb{R}$ is the *memory kernel.*

The parameter $\delta \in \mathbb{R}$ is interpreted as an initial input to the jump rate function.

Simple Erlang kernel

Assumption 1

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$$h(t) = ce^{-\alpha t}\frac{t^n}{n!}, \ t \geq 0,$$

where $c \in \mathbb{R}$, $\alpha > 0$ and $n \in \mathbb{N}$.

For each $0 \leq k \leq n$, consider, for each $t \geq 0$,

$$X_t^{(k)} = \delta + \int_{]0,t]} ce^{-\alpha(t-s)} \frac{(t-s)^{(n-k)}}{(n-k)!} dN_s, \tag{2}$$

in this case,

$$\lambda_t = f\left(X_{t-}^{(0)}\right),$$

and, for $t \geq 0$,

Associated PDMP process

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with initial condition $X_0^{(k)} = x_0^{(k)} = \int_{]-\infty,0]} ce^{\alpha s} \frac{(-s)^{(n-k)}}{(n-k)!} n(ds)$.

The associated PDMP is the Markov process $X = (X_t)_{t\geq 0}$ taking values in $\mathbb{R}^n$, defined, for each $t \geq 0$, by

$$X_t = \left(X_t^{(0)}, \ldots, X_t^{(n)}\right).$$

We call the process X *Markovian cascade of memory terms*.

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$$\mathcal{L}g(x) = \langle F(x), \nabla g(x) \rangle + f\left(x^{(0)}\right)\left(g\left(x + ce_{(n)}\right) - g(x)\right),$$

where $x = (x^{(0)}, \dots, x^{(n)})$ and $e_{(n)} \in \mathbb{R}^n$ is the unit vector having entry 1 in the coordinate n and 0 elsewhere.

Finally, $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the vector field associated to the system of ODE's

$$\begin{cases} \frac{d}{dt} x_t^{(0)} = x_t^{(1)} - \alpha x_t^{(0)} \\ \vdots \\ \frac{d}{dt} x_t^{(n-1)} = x_t^{(n)} - \alpha x_t^{(n-1)} \\ \frac{d}{dt} x_t^{(n)} = -\alpha x_t^{(n)} \end{cases}$$

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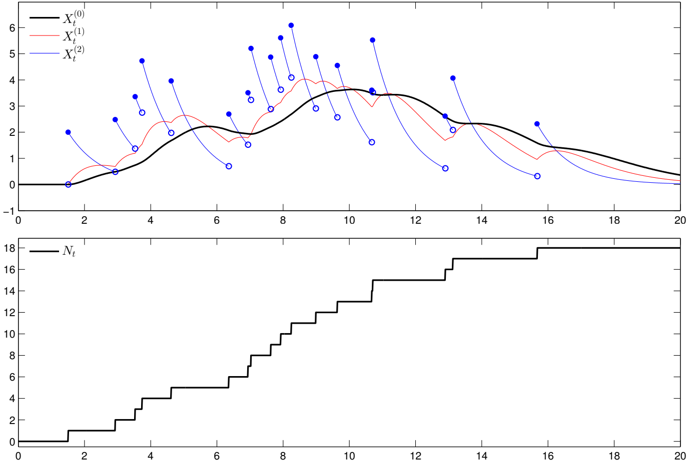
- Jumps introduce discontinuities only in the coordinates $X_t^{(n)}$ of X_t.



A finite joint realization of the Markovian cascade $X = (X_t)_{0 \le t \le T}$ (upper panel) and its associated counting process $N = (N_t)_{0 \le t \le T}$ (lower panel) for the choices $n = 2$, $c = 2$, $\alpha = 1$, $T = 20$ and $f(x) = x/5 + 1$ with initial input $x_0 = (x_0^{(0)}, x_0^{(1)}, x_0^{(2)}) = (0, 0, 0)$. The blue (resp. red and black) trajectory corresponds to the realisation of $(X_t^{(2)})_{0 \le t \le T}$ (resp. $(X_t^{(1)})_{0 \le t \le T}$ and $(X_t^{(0)})_{0 \le t \le T}$).

Sum of Erlang kernels

Consider the memory kernel $h : \mathbb{R}_+ \to \mathbb{R}$ can be written as Erlang kernel

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where $c \in \mathbb{R}$, $\alpha > 0$ and $n \in \mathbb{N}$.

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$$h(t) = \sum_{i=1}^{L} c_i e^{-\alpha_i t} \frac{t^{n_i}}{n_i!}, \quad t \geq 0, \tag{4}$$

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Any Hawkes process N having integrable memory kernel h can be approximated by a sequence of Hawkes processes $N^{(n)}$ having Erlang memory kernel $h^{(n)}$ such that $\|h^{(n)} - h\|_{L^1(\mathbb{R}_+)} \to 0$ as $n \to \infty$ and

$$E\|N - N^{(n)}\|_t \leq C_T \int_0^t |h^{(n)} - h|(s)ds,$$

for all $t \leq T$, where $\|N - N^{(n)}\|_t$ denotes the total variation distance between N and $N^{(n)}$ on $[0, t]$.

Associated PDMP process

Write $\kappa = L + \sum_{i=1}^{L} n_i$. The associated PDMP $X = (X_t)_{t \geq 0}$ taking values in $\mathbb{R}^\kappa$, defined, for each $t \geq 0$, by

$$X_t = \left(X_t^{(1)}, \ldots, X_t^{(L)}\right) \text{ with } X_t^{(i)} = \left(X_t^{(i,0)}, \ldots, X_t^{(i,n_i)}\right),\ 1 \leq i \leq L. \quad (5)$$

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where $x = \left(x^{(1)}, \dots, x^{(L)}\right)$ with $x^{(i)} = (x^{(i,0)}, \dots, x^{(i,n_i)})$ and $e_{(i,n_i)} \in \mathbb{R}^\kappa$ is the unit vector having entry 1 in the coordinate (i, n_i), and 0 elsewhere.

And $F : \mathbb{R}^\kappa \mapsto \mathbb{R}^\kappa$ is the vector field associated to the system of first-order ODE's

$$\begin{cases} \frac{d}{dt} x_t^{(i,0)} = x_t^{(i,1)} - \alpha_i x_t^{(i,0)} \\ \vdots \\ \frac{d}{dt} x_t^{(i,n_i-1)} = x_t^{(i,n_i)} - \alpha_i x_t^{(i,n_i-1)} \\ \frac{d}{dt} x_t^{(i,n_i)} = -\alpha x_t^{(i,n_i)}, \ 1 \leq i \leq L, \end{cases} \tag{7}$$

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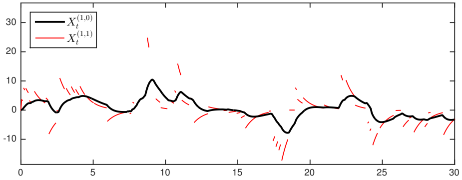
$X_t^{(1,0)}$
$X_t^{(1,1)}$
30
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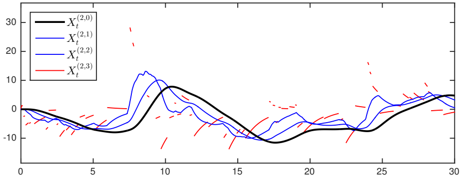
$X_t^{(2,0)}$
$X_t^{(2,1)}$
$X_t^{(2,2)}$
$X_t^{(2,3)}$
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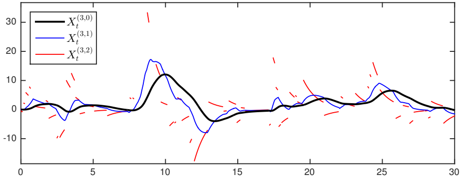
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Random jump rates

Consider the Markov process $X = (X_t)_{t\geq 0}$ taking values in $\mathbb{R}^\kappa$ whose generator is given, for any smooth and bounded function $g : \mathbb{R}^\kappa \mapsto \mathbb{R}$, by

$$\mathcal{L}g(x) = \langle F(x), \nabla g(x)\rangle + f\Big(\sum_{i=1}^{L} x^{(i,0)}\Big)\Big(g\big(x + \sum_{i=1}^{L} c_i e_{(i,n_i)}\big) - g(x)\Big),$$

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where $F : \mathbb{R}^\kappa \mapsto \mathbb{R}^\kappa$ is the vector field associated to the system (7) and $G(dc_1, \ldots, dc_L)$ is a probability measure on $\mathbb{R}^L$.

Well-posedness of the process

Assumption 2

The probability measure G on $\mathbb{R}^L$ has finite first moments, i.e.,

$$\int \sum_{i=1}^{L} |c_i| G(dc_1, \ldots, dc_L) < \infty.$$

Proposition

Under Assumptions 1 and 2. Let $N = (N_t)_{t \geq 0}$ be the counting process associated to the jumps of the Markov Process $X = (X_t)_{t \geq 0}$ having generator given by (7), starting from $x_0 \in \mathbb{R}^\kappa$. Then N has P_{x_0}-almost surely a finite number of jumps on each interval $[s, t], 0 \leq s < t < \infty$.

Let μ and ν be two probability measures on $\mathbb{R}^\kappa$. The Wasserstein distance between μ and ν is defined by

$$W_1(\mu,\nu) = \inf\left\{ \int_{\mathbb{R}^\kappa}\int_{\mathbb{R}^\kappa} \|x-y\|_1 \gamma(dx,dy), \gamma \in \Gamma(\mu,\nu) \right\}. \tag{9}$$

$(P_t)_{t\geq 0}$ for the transition semigroup of the process X with generator (7).

Theorem 1 (for $\alpha_i \equiv \alpha > 1,\ \forall i$)

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Theorem 1 (for $\alpha_i \equiv \alpha > 1, \ \forall i$)

Suppose f not bounded but only Lipschitz continuous, we suppose moreover that

$$\|f\|_{Lip}\Big(\int \sum_{i=1}^{L} \frac{1}{\alpha^{n_i}} |c_i| G(dc_1,\ldots,dc_L)\Big) < \alpha. \tag{10}$$

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$$\|f\|_{Lip}\left(\int \sum_{i=1}^{L} \frac{1}{\alpha^{n_i}} |c_i| G(dc_1,\dots,dc_L)\right) < \alpha. \quad (10)$$

Then, there exists an unique invariant probability measure π of the process X such that for any probability measure ν on $\mathcal{B}(\mathbb{R}^\kappa)$,

$$W_1(\pi,\nu P_t) \leq Ce^{-dt} W_1(\pi,\nu).$$

for $C > 0$ and $d > 0$ properly chosen.

Definition 2

The process $(X_t)_{t \geq 0}$ is said to be *recurrent in the sense of Harris* if there exists a sigma-finite measure m on $\mathcal{B}(\mathbb{R}^\kappa)$ such that $m(A) > 0$ implies that for all $x \in \mathbb{R}^\kappa$, P_x-almost surely,

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Theorem 2

Grant Assumptions 1 and 2. Suppose moreover that (9) holds and that $G(dc_1, \ldots, dc_L) = \prod_{i=1}^L G_i(dc_i)$ for probability measures G_i on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying supp $(G_i) \cap \{0\}^c \neq \emptyset$, for all $1 \leq i \leq L$. Finally, suppose that f is lower bounded.

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1. Then $(X_t)_{t\geq 0}$ is positive Harris recurrent with unique invariant measure $\pi(dx)$.

2. Let $\bar{X}_t$ be a stationary version of the process and suppose that $(X_t)_{t\geq 0}$ starts from $X_0 = x_0 \in \mathbb{R}^\kappa$, both evolving according to (7). Then $\bar{X}$ and X couple almost surely in finite time.

Simulation

We write $\varphi_t(x) = (\varphi_t^{(1)}(x), \dots, \varphi_t^{(L)}(x))$ for the unique solution, starting from $x \in \mathbb{R}^\kappa$, of the system of ODE's (7)

Denote $\|x\|_\infty = \max\{|x^{(i,k)}|, 1 \le i \le L, 0 \le k \le n_i\}$ and $n = \max_{1 \le i \le L} n_i$

Proposition

For each $x \in \mathbb{R}^\kappa$, let $M(x) = \max\{|\varphi_t^{(i,0)}(x)| : 1 \le i \le L, t \ge 0\}$

$$M(x) \le e\|x\|_\infty \Big(1 \vee \Big(\frac{n}{\alpha e}\Big)^n\Big).$$

Define the function $\mathbb{R}^\kappa \in x \mapsto f^*(x)$ by

$$f^*(x) = \left\{ \begin{array}{ll} \max\{f(y) : y \in [0, LM(x)]\}, & \text{if } x \in \mathbb{R}^\kappa_+ \\ \max\{f(y) : y \in [-LM(x), 0]\}, & \text{if } x \in \mathbb{R}^\kappa_- \\ \max\{f(y) : y \in [-LM(x), LM(x)]\}, & \text{else} \end{array} \right\}.$$

Simulation algorithm

Let $T_0 = 0$ and $(T_k)_{k\geq 1}$ denote the sequence of jump times of the Markovian cascade X

- Draw an exponential random variable τ with parameter $f^*(x)$
- Draw a uniform random variable U on $[0, 1]$.
- If $U \leq f(\sum_{i=1}^{L} \varphi_{T_k+\tau}^{(i,0)}(x))/f^*(x)$, then define the next jump time $T_{k+1} = T_k + \tau$.
- If not, repeat this procedure starting from $X_{T_k+\tau} = \varphi_\tau(x)$.

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It provides an exact simulation of the Markovian cascade X. No approximation procedure is required.

Thanks